Discrete and intersample analysis of sampled-data systems with non-uniform sampling

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Classical control loop

Ideal Hypothesis :
- Sampling and actuation are periodic and synchronous
Motivating problems: Digital control

Classical control loop

Real-time problem: the system is affected by timing problems
  ▶ sampling jitter (sensor, multitasking processors, packet dropouts in communication channels)
  ▶ unknown time varying delays (not addressed here)

(Wittenmark, Nilsson, Torngren, 1995)
Consider the system

\[ \dot{x}(t) = Ax(t) + Bu(t), \ \forall t \in \mathbb{R}^+ \]

with a sampled-data control:

\[ u(t) = Kx(t_k), \ \forall t \in [t_k, t_{k+1}) \]

**Problem**: Is the system stable under sampling variations?
Sampling jitter example (Zhang, 2001)

\[ \dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \quad h_k \in \{T_1, T_2\} \]

\[ A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = -\begin{bmatrix} 1 & 6 \end{bmatrix} \]
Sampling jitter example (Zhang, 2001) ⇒ instability

\[ \dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \quad h_k \in \{ T_1, T_2 \} \]

\[ A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = -[1 \quad 6] \]

Open problem: provide tools for robust stability and performance analysis!
Discrete-time model

\[ \dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \quad h_k \in [h_{\min}, h_{\max}] \]

\[ x(t_{k+1}) = e^{(t_{k+1}-t_k)A} x(t_k) + \int_{0}^{(t_{k+1}-t_k)} e^{sA} ds Bu(t_{k+1}) \]

\[ \Rightarrow \quad x_{k+1} = \Lambda(h_k) x(t_k) \]
Discrete-time model: difference inclusion

Continuous-time model

\[ \dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \quad h_k \in [h_{\text{min}}, h_{\text{max}}] \]

Equivalent discrete-time model

\[ x^+ \in \mathcal{H}(x), \quad \mathcal{H}(x) = \{ y : y = \Lambda(h)x, \quad h \in [h_{\text{min}}, h_{\text{max}}] \} \]

with

\[ \Lambda(h) = e^{hA} + \int_0^h e^{sA} dsBK \]

(Continuous-time system stable iff Difference inclusion is stable)

Problems for stability analysis:

- How to deal with the uncertain matrices with exponential form?
  \[ \int_0^h e^{sA} ds \]

- Which class of Lyapunov functions should we chose?
Exponential uncertainty

Integration operator

\[ \Lambda(h) = e^{hA} + \int_0^h e^{sA} dsBK, \quad h \in [h_{\text{min}}, h_{\text{max}}] \]

For the case of quadratic Lyapunov functions \( V(x) = x^T P x \):

\[ P > 0, \quad \Lambda^T(h) P \Lambda(h) - P < 0, \quad h \in [h_{\text{min}}, h_{\text{max}}] \]

Problem: infinite number of stability conditions
Exponential uncertainty

\[ \Lambda(\rho) = e^{\rho A} + \int_0^\rho e^{sA} dsBK = I + \int_0^\rho e^{sA} ds \ (A + BK) \]

\[ \Gamma(\rho) = \int_0^\rho e^{As} ds, \ h_{\text{min}} < \rho < h_{\text{max}} \]

curve in the space of \( \mathbb{R}^{n \times n} \) matrices
Consider a gridding on the space of parameters $\rho$

$$\forall \rho \in \{\rho_1, \rho_2, \ldots, \rho_N\} \subset [h_{\text{min}}, h_{\text{max}}].$$

- Finite number of conditions:
  $$P > 0, \quad \Lambda^T(\rho_i) P \Lambda(\rho_i) - P < 0,$$
  $$i = 1, \ldots, N$$

- Simple for illustration
- Approximative solution
Exponential uncertainty - Ellipsoidal embedding

\[ \Gamma(\rho) = \Gamma_0 + \Delta \Gamma \]
\[ \Delta \Gamma^T \Delta \Gamma < \epsilon I \]

- LMI solution can be obtained (Gahinet, IEEE TAC, 1994), (Fujioka, IEEE TAC, 2009)
- Exact solutions
- Conservatism due to over-approximation.
Exponential uncertainty - Polytopic Embedding

$$\exists \mu_i > 0, \ \forall i = 1, \ldots, N$$

s.t. $$\sum_{i=1}^{N} \mu_i = 1,$$

$$\Gamma(\rho) = \sum_{i=1}^{N} \mu_i A_i$$

(Olaru, Niculescu, IFAC World Congress 2007),
(Cloosterman, et. al, Trans. Autom. Contr. 2009),
Jordan normal form (Olaru, 2006), (Cloosterman, 2009)

For \( A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \)

\( \Gamma(\rho) = \begin{pmatrix} \alpha_1(\rho) & 0 \\ 0 & \alpha_2(\rho) \end{pmatrix} \)

with

\[ \alpha_i(\rho) = \int_0^\rho e^{\lambda_i s} \, ds \]

vertex = max or min \( \alpha_i(\rho) \)

\( \lambda_1 = -1.5, \lambda_2 = 0.2 \)

\( \Gamma(\rho) = \sum_{j=0}^{2^n} \mu_j A_j \)
Polytopic embedding + gridding

For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\Gamma(\rho) = \begin{pmatrix} \alpha_1(\rho) & 0 \\ 0 & \alpha_2(\rho) \end{pmatrix}$$

with

$$\alpha_i(\rho) = \int_0^\rho e^{\lambda_i s} \, ds$$

vertex = max or min $\alpha_i(\rho)$

$$\lambda_1 = -1.5, \lambda_2 = 0.2$$

$$\Gamma(\rho) = \sum_{j=0}^{5 \times 2^n} \mu_j A_j$$
Tractable LMI conditions

\[
\sum_{j=1}^{N} \mu_j = 1, \mu_j > 0, \forall j = 1, \ldots, N,
\]

\[
\Gamma(\rho) = \sum_{i=1}^{N} \mu_j A_j,
\]

- \( \Lambda(\rho) = I + \Gamma(\rho)(A + BK) \in co\{I + A_j(A + BK), j = 1, \ldots, N\} \)

\[
P = P^T > 0
\]

\[
(I + A_j(A + BK))^T P (I + A_j(A + BK)) - P < 0, \forall j = 1, \ldots, N
\]

- Finite number of LMI stability conditions!
Remarks about polytopic embeddings

- A polytopic representation with less vertex may be obtained based on Taylor series approximation
  
  \[ \text{(Hetel, Daafouz, Iung, TAC, 2006)} \]
- The conservatism due to the use of an embedding may be tunned according to the desired numerical complexity
- Allow to use more efficient Lyapunov functions.
Lyapunov functions

Equivalent Difference inclusion

\[ x^+ \in \mathcal{H}(x), \quad \mathcal{H}(x) = \{ y : y = \Lambda(h)x, \ h \in T = [h_{\text{min}}, h_{\text{max}}] \} , \]

with

\[ \Lambda(h) = e^{hA} + \int_0^h e^{sA} dsBK \]

General Remarks on Linear Difference Inclusions (LDI) :

- Quadratic Lyapunov Functions (QLF) \( V(x) = x^T P x \) are sufficient only for stability (not necessary)

- There are cases of LDI which are stable for which no QLF exists

(Dayawansa, Martin, IEEE TAC, 1999)
Lyapunov functions

Equivalent Difference inclusion

\[ x^+ \in \mathcal{H}(x), \quad \mathcal{H}(x) = \{ y : y = \Lambda(h)x, \ h \in T = [h_{\text{min}}, h_{\text{max}}] \} , \]

General Remarks on Linear Difference Inclusions (LDI):

- Necessary and Sufficient stability conditions: the existence of Quasi-Quadratic Lyapunov functions

\[ V(x) = x^T \mathcal{L}[x] x, \]

\[ \mathcal{L}[x] = \mathcal{L}^T[x] = \mathcal{L}[ax], \quad \forall x \neq 0, \ a \in \mathbb{R}, \ a \neq 0 \]

s.t. the following relation is satisfied:

\[ V(x) - \max_{h \in T} V(\Lambda(h)x) > 0. \]

(Molchanov and Pyatniskii, SCL 1989)
Quasi-quadratic Lyapunov functions

For \( V(x) = x^T L_{[x]} x \) and polytopic LDI

- BMI criteria based on composite quadratic approximations
  \[
  V_c(x) = \max_{i=1,\ldots,p} x^T L_i x, \quad L_i = L_i^T > 0.
  \]
  (Hu, Blanchini, Automatica, 2010)

- level set of composite function \( V_c(x) \) = intersection of ellipsoids

- no LMI existence criteria in the literature
Stability based on non-monotonous functions

(Megretzki, IEEE CDC 1994); (Krusezwski, Guerra, IEEE TAC 2008)

Classical approach
\[ \forall x_k, \ V(x_{k+1}) - V(x_k) < 0 \]

New approach
\[ \forall x_k, \ V(x_{k+\alpha}) - V(x_k) < 0 \]
Stability based on non-monotonous functions

Properties:

- $\alpha = 1$ - case of classical Lyapunov functions
- A LDI is stable iff there exist a finite $\alpha \in \mathbb{N}$ such as $V(x_{k+\alpha}) < V(x_k)$. 
Stability based on non-monotonous functions

\[ x^+ \in \mathcal{H}(x), \quad \mathcal{H}(x) = \{ y : y = \Lambda(h)x, \ h \in T = [h_{\text{min}}, h_{\text{max}}] \}, \]

Denote

\[ \sigma = \{ h^i \}_{i=0}^{\alpha-1} \] sequence of \( \alpha \) sampling times and

\[ \Phi_{\sigma}(\alpha) \] transition matrix associated to \( \sigma \):

\[
\Phi_{\sigma}(\alpha) = \begin{cases} 
\Lambda(h^{\alpha-1}) \ldots \Lambda(h^1) \Lambda(h^0), & \alpha > 0 \\
I, & \alpha = 0.
\end{cases}
\]

with the function \( V(x) = x^T Px \).

**Proposition**: The equilibrium point \( x = 0 \) is asymptotically stable iff there exists a finite \( \alpha \in \mathbb{N}^+ \) s.t.

\[ \Phi_{\sigma}^T(\alpha)P\Phi_{\sigma}(\alpha) - P < 0 \]

for all \( \alpha \) length sequences with values in \( T \).
Stability based on non-monotonous functions

\[ \Lambda(h) \in coZ \text{ where } Z = \{Z_1, Z_2, \ldots, Z_N\} \]
Consider the set of products of \( \alpha \) matrices with values in \( Z \)

\[ \mathcal{Y}_\alpha(Z) = \{Y : Y = \prod_{i=0}^{\alpha-1} Z_{\mu_i}, Z_{\mu_i} \in Z\} \cdot \]

**Proposition.** If there exist a positive integer \( \alpha \) and a matrix \( P = P^T \succ 0 \) that satisfy

\[ P \succ Y^T PY, \quad \forall Y \in \mathcal{Y}_\alpha(Z), \]

then the equilibrium point \( x = 0 \) of the LDI is asymptotically stable.
Stability based on non-monotonous functions

- finite number of LMI (complexity to be tuned according to number of vertex $N$ and the horizon of analysis $\alpha$)
- is there any relation with Quasi-Quadratic Lyapunov functions?
- is there a constructive manner for obtaining a Quasi-Quadratic Lyapunov functions using $P$ ?
Relation between the two approaches

**Proposition.** If there exist a positive integer $\alpha$ and a matrix $P = P^T \succ 0$ that satisfy

$$P \succ Y^T PY, \ \forall Y \in \mathcal{Y}_\alpha(\mathcal{Z}),$$

then there exists a composite quadratic Lyapunov function for the LDI,

$$V_c(x) = \max_{i=1,\ldots,M} x^T L_i x \text{ s.t. } V_c(x) > \max_{\theta \in \mathcal{T}} V_c(\Lambda(\theta)x)$$

where $L_i, \ i = 1, \ldots, M = N^{\alpha-1}$ are an enumeration of the elements in the set

$$\Omega = \left\{ Q^Z_\sigma(\alpha) = \sum_{j=1}^{\alpha-1} \left( \prod_{r=1}^{j} Z_{\mu_r} \right)^T P \left( \prod_{r=1}^{j} Z_{\mu_r} \right) + P, \ \sigma = \{\mu_r\}_{r=1}^{N-1} \in \{1, \ldots, N\}^\alpha \right\}.$$
Numerical examples

\[ A_c = \begin{pmatrix} -0.5 & 0 \\ 0 & 3.5 \end{pmatrix}, \quad B_c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 1.02 & -5.62 \end{pmatrix}. \]

- \( \Lambda(h) \) is Schur for any sampling interval \( h \in [0, 0.46] \).
- \( \Phi = (\Lambda(0.1))^6 \Lambda(0.43) \) is not Schur (exists a periodic unstable sequence).
- \( h_k \in \{0.1, h_{\text{max}}\} \)
- Exists a QLF for \( h_{\text{max}} = 0.36 \).
- with \( \alpha = 7, h_{\text{max}} = 0.41 \).
Numerical examples (Zhang, 2001)

\[ A_c = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} \text{ and } K = (-3.75 \quad 11.5). \]

- \( \Lambda(h) \) is Schur for any sampling interval \( h \in [0, 0.172] \).
- \( h_k \in [0.01, h_{\text{max}}] \)
- for (Mirkin, 2007), (Fridman, 2004), (Hespanha, 2008) \( h_{\text{max}} < 1.36 \)
- with \( \alpha = 1 \), Taylor convex embedding, \( h_{\text{max}} = 0.17 \).
Numerical examples (Dayawansa, Martin, 1999)

Sampled-data version of

\[ \dot{x} = A_\sigma x, \quad \sigma \in \{1, 2\} \]

\[ A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -8 \\ 1/8 & -1 \end{pmatrix} \]

No QLP can be found
Numerical examples (Dayawansa, Martin, 1999)

Sampled-data version of

\[
\dot{x} = A_\sigma x, \quad \sigma \in \{1, 2\}
\]

\[
A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -8 \\ 1/8 & -1 \end{pmatrix}
\]

No QLP can be found

Positive Invariant Set based on Non-monotonous functions
Continuous-time approaches

Main Discrete-time problem: inter-sampling behavior

Evolution of a Lyapunov function $V(x) = x^T P x$

- Strictly decreasing at $t = t_k$
  (sufficient condition for stability analysis)
- Increasing in between the sampling times
  (false evaluation of control performance)
Existing work - Continuous-time: time delay approach

\[ u(t) = Kx(t_k) = Kx(t - \tau) \text{ with } \tau = t - t_k, \ 0 < \tau_k < h_{max} \]
Existing work - Continuous-time

- Fridman et al, 2004 (input delay approach)
- Mirkin, 2007 (robust control equivalent)
- Hespanha, 2008 (impulsive delay diff. eq.)

Advantage:
- Directly extend to performance study (decay rate)
- Take into account the inter-sampling behavior

Inconvenient:
- Do not take into account the sawtooth form of the delay (conservatism)

Note: discrete-time approaches implicitly take into account this aspect via the integration operator
Goal

Provide a continuous-time method that takes into account the sawtooth form of the delay (advantage of discrete-time methods for conservatism reduction)
Case of quadratic Lyapunov functions

For \( \frac{dx(t)}{dt} = Ax(t) + BKx(t - \tau(t)), \quad \tau(t) := t - t_k, \quad \forall t \in [t_k, t_{k+1}) \)

and \( V(x) = x^T P x \)

- Derivative of Lyapunov function

\[
\frac{dV(x)}{dt} = 2x^T(t)P(Ax(t) + BKx(t - \tau)) .
\]

- Sawtooth evolution of delay can be introduced by using the integration operator \( \Lambda(\cdot) \) used for the discrete-time model:

\[
x(t) = \Lambda(t - t_k)x(t_k), \quad \forall t \in [t_k, t_{k+1})
\]
Case of quadratic Lyapunov functions

\[
\frac{dV(x)}{dt} = 2x^T(t)P(Ax(t) + BKx(t - \tau)) < -2\alpha V(x(t)).
\]

and

\[
x(t) = \Lambda(\tau)x(t_k), \forall \tau \in [0, h_{max}]
\]

**Proposition:** If \( \exists P = P^T \succ 0 \ G_1, G_2 \) s.t.

\[
\begin{pmatrix}
A^TP + PA + G_1 + G_1^T + \alpha P & PBK - G_1\Lambda(\tau) + G_2^T \\
K^TB^TP - \Lambda^T(\tau)G_1^T + G_2 & -G_2\Lambda(\tau) - \Lambda^T(\tau)G_2^T
\end{pmatrix} \prec 0,
\]

\( \forall \tau \in [0, h_{max}] \) then

\[
\frac{dV(x)}{dt} < -2\alpha V(x), \forall x \neq 0
\]

**Remark:** satisfied if \( \Lambda(\tau) \) is non-singular for all \( \tau \in [0, h_{max}] \)
Case of quadratic Lyapunov functions

Let $\Lambda(\tau) \in \text{co}\{A_j\}_{j=1}^N$, $\forall \tau \in [0, h_{\text{max}}]$. 

**Proposition**: If $\exists P = P^T \succ 0$ $G_1$, $G_2$ s.t.

$$
\begin{bmatrix}
A^T P + PA + G_1 + G_1^T + \alpha P & PBK - G_1 A_j + G_2^T \\
K^T B^T P - A_j^T G_1^T + G_2 & -G_2 A_j - A_j^T G_2^T
\end{bmatrix} \prec 0,
\forall l = 1, \ldots, N.
$$

then

$$
\frac{dV(x)}{dt} < -2\alpha V(x), \forall x \neq 0
$$

Finite number of conditions may be obtained using the polytopic convex embedding for $\Lambda(\tau)$. 
Non-singularity and Quasi-Quadratic Lyapunov functions

**Remark**: Existence of functions of the class $V_c(x)$ is necessary when $\Lambda$ is non-singular

$$\frac{dx(t)}{dt} = Ax(t) + BKx(t_k), \forall t \in [t_k, t_{k+1}),$$

$$x(t_k) = \Lambda^{-1}(\tau)x(t)$$

$$\frac{dx}{dt} \in \mathcal{H}_c(x), \mathcal{H}_c(x) = \{(A + BK\Lambda^{-1}(\tau))x, \tau \in [0, h_{max}]\},$$

based on (Molchanov, Pyatniski, 1989), (Hu, Blanchini, Automatica 2010)
Extension: Quasi-Quadratic Lyapunov functions

\[
\frac{dx}{dt} \in G_c(x, t), \quad G_c(x, t) = \{(A + BK)x, Ax + BKx(t_k)\},
\]

For

\[
V_c(x) = \max_{i=1,...,M} x^T L_i x
\]

using

\[
\max_{y(t) \in G_c(x,t)} \nabla y(t) V_c(x(t)) = 2 \frac{dx^T}{dt} L_i x, \quad \text{for } x^T (L_j - L_i)x < 0
\]

we obtain the following set of conditions

\[
\begin{pmatrix}
A^TL_i + L_iA + \lambda L_i - \sum_{i \neq j} \beta_{ij} (L_j - L_i) + G_1 + G_1^T \\
K^TB^TL_i - \Lambda^T(\tau)G_1^T + G_2
\end{pmatrix}
\begin{pmatrix}
L_iBK - G_1\Lambda(\tau) + G_2^T \\
-G_2\Lambda(\tau) - \Lambda^T(\tau)G_2^T
\end{pmatrix} < 0,
\]

\[i, j = 1, \ldots, M, \forall \tau \in [0, h_{\max}].\]
Consider the quadratic function
\[ V(x) = x^T P x, \quad P = P^T \succ 0. \]

Asymptotic stability conditions:
\[ \dot{V}(x(t)) < 0 \text{ whenever } V(x(t_k)) < \alpha V(x(t)), \text{ with } \alpha > 1 \]

Matrix Inequalities conditions:
\[ P = P^T \succ 0, \text{ a scalar } \epsilon > 0, \text{ and matrices } G_1, G_2 \in \mathbb{R}^{n \times n} \text{ s.t.} \]
\[ \left( \begin{array}{cc}
A^T P + PA + \epsilon \alpha P + G_1 + G_1^T & PBK - G_1 \Lambda(\theta) + G_2^T \\
K^T B^T P - \Lambda^T(\theta) G_1^T + G_2 & -G_2 \Lambda(\theta) - \Lambda^T(\theta) G_2^T - \epsilon P
\end{array} \right) \prec 0. \]
\[ \forall \theta \in [0, \theta_{\text{max}}] \]
Numerical example

Consider a continuous-time system described by:

\[
A = \begin{pmatrix} 1 & 15 \\ -15 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\
1 \end{pmatrix}.
\]

\(\lambda(A) = 1 \pm 15i\)

\(K\) - obtained by pole placement: \(\lambda(A + BK) = -1 \pm i\)

Stability analysis comparison:

\(\quad \) (Mirkin, TAC 2007): \(h \in [0, 0.014]\)

\(\quad \) (Naghshtabarzi, Hespanha, Teel, SCL 2008): \(h \in [0, 0.033]\)

\(\quad \) (Fujioka, Automatica 2009): \(h \in [0, 0.07]\)

\(\quad \) continuous-time + polytopic embedding: \(h \in [0, 0.09]\)

(singularity for 0.092)

\(\quad \) Lyapunov-Razumikhin + polytopic embedding: \(h \in [0, 0.14]\)

\(\quad \) discrete-time approach: \(h \in [0.01, 0.15]\)
Numerical example
Conclusion and Perspective

- Robustness to sampling jitter
- Provide robust methods for stability
- Show how to reduce the conservatism of stability analysis by taking into account the sawtooth form of the delay
- Perspective: control design