Discrete and intersample analysis of sampled-data systems with non-uniform sampling

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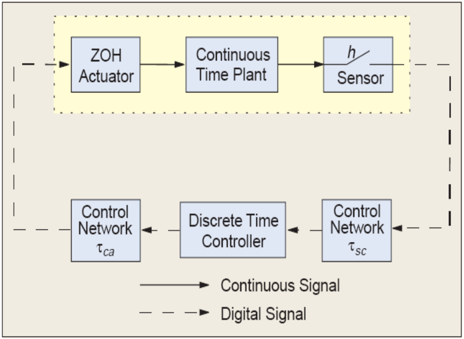
Outline

Motivating problems : Digital control

Classical control loop

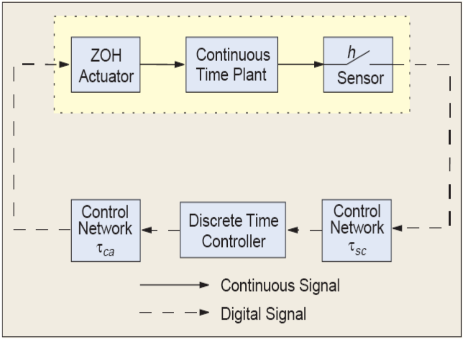


Ideal Hypothesis :

- Sampling and actuation are periodic and synchronous

Motivating problems : Digital control

Classical control loop

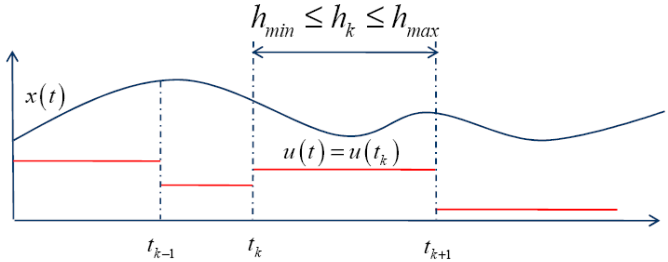


Real-time problem : the system is affected by timing problems

- sampling jitter (sensor, multitasking processors, packet dropouts in communication channels)
- unknown time varying delays (not addressed here)

(Wittenmark, Nilsson, Torngren, 1995)

Problem Formulation



Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \ \forall t \in \mathbb{R}^+$$

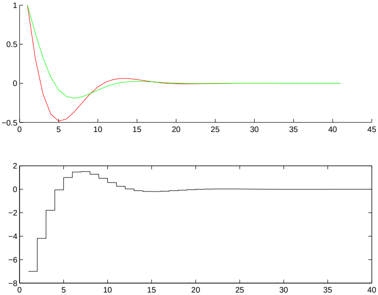
with a sampled-data control :

$$u(t) = Kx(t_k), \ \forall t \in [t_k, t_{k+1})$$

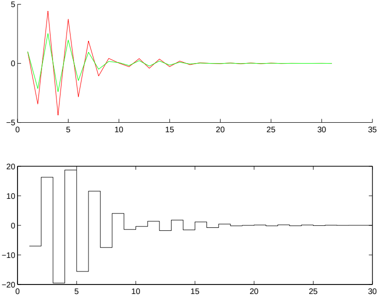
Problem : Is the system stable under sampling variations ?

Sampling jitter example (Zhang,2001)

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \; h_k \in \{T_1, T_2\}$$



$h = 0.18s$

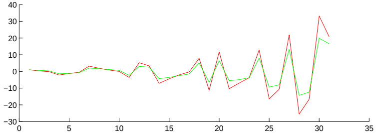


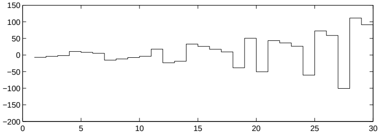
$h = 0.58s$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}; \quad K = -[1 \;\; 6]$$

Sampling jitter example (Zhang,2001) $\Rightarrow$ instability

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \;\; h_k \in \{T_1, T_2\}$$



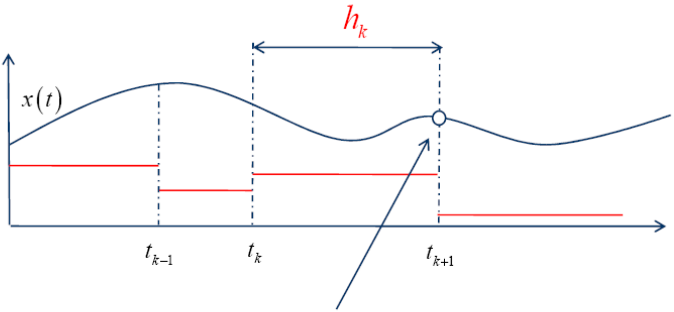


$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}; \quad K = -[1 \;\; 6]$$

Open problem : provide tools for robust stability and performance analyzis !

Discrete-time model

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \; h_k \in [h_{min}, h_{max}]$$



$$x(t_{k+1}) = e^{(t_{k+1}-t_k)A} x(t_k) + \int_0^{(t_{k+1}-t_k)} e^{sA} ds B u(t_{k+1})$$

$$\Longrightarrow \quad x_{k+1} = \Lambda(h_k) x(t_k)$$

Discrete-time model : difference inclusion

Continuous-time model

$$\dot{x} = Ax + Bu_k, \quad u_k = Kx_k, \; h_k \in [h_{min}, h_{max}]$$

Equivalent discrete-time model

$$x^+ \in \mathcal{H}(x), \; \mathcal{H}(x) = \{y \;:\; y = \Lambda(h)x, \; h \in [h_{min}, h_{max}]\},$$

with

$$\Lambda(h) = e^{hA} + \int_0^h e^{sA} ds BK$$

(Continuous-time system stable iff Difference inclusion is stable)

Problems for stability analysis :

- how to deal with the uncertain matrices with exponential form ?

$$\int_0^h e^{sA} ds$$

- which class of Lyapunov functions should we chose ?

Exponential uncertainty

- Integration operator

$$\Lambda(h) = e^{hA} + \int_0^h e^{sA} ds BK, \;\; h \in [h_{min}, h_{max}]$$

- For the case of quadratic Lyapunov functions $V(x) = x^T P x$:

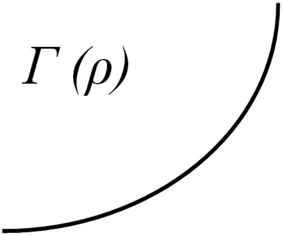
$$P > 0, \quad \Lambda^T(h) P \Lambda(h) - P < 0, \;\; h \in [h_{min}, h_{max}]$$

Problem : infinite number of stability conditions

Exponential uncertainty

$$\Lambda(\rho) = e^{\rho A} + \int_0^\rho e^{sA} ds BK = I + \int_0^\rho e^{sA} ds \, (A + BK)$$

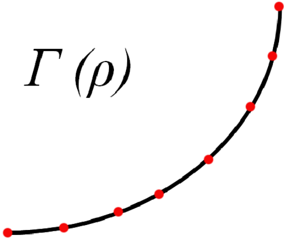
$$\Gamma(\rho) = \int_0^\rho e^{As} ds, \;\; h_{min} < \rho < h_{max}$$



curve in the space of $\mathbb{R}^{n\times n}$ matrices

Exponential uncertainty : Gridding

(Sala, Automatica, 2004)



Consider a gridd on the space of parameters ρ

$$\forall \rho \in \{\rho_1, \rho_2, \ldots, \rho_N\} \subset [h_{min}, h_{max}].$$

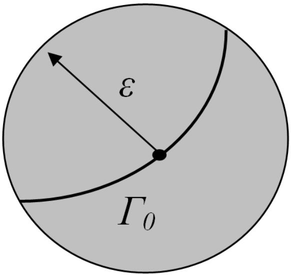
- Finite number of conditions :

  $$P > 0, \;\; \Lambda^T(\rho_i) P \Lambda(\rho_i) - P < 0,$$

  $$i = 1, \ldots, N$$

- Simple for illustration
- Approximative solution

Exponential uncertainty - Ellipsoidal embedding

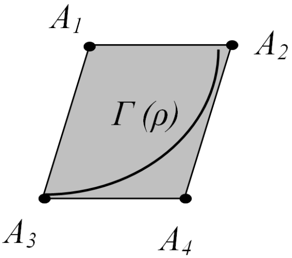


$$\Gamma(\rho) = \Gamma_0 + \Delta\Gamma$$

$$\Delta\Gamma^T \Delta\Gamma < \epsilon I$$

- LMI solution can be obtained (Gahinet, IEEE TAC, 1994), (Fujioka, IEEE TAC, 2009)
- Exact solutions
- Conservatism due to over-approximation.

Exponential uncertainty - Polytopic Embedding



$$\exists \mu_i > 0,\ \forall i = 1, \dots, N$$

$$\text{s.t.} \sum_{i=1}^{N} \mu_i = 1,$$

$$\Gamma(\rho) = \sum_{i=1}^{N} \mu_i A_i$$

(Hetel, Daafouz, Iung, Trans. Autom. Contr. 2006)
(Olaru, Niculescu, IFAC World Congress 2007),
(Cloosterman, et. al, Trans. Autom. Contr. 2009),

Jordan normal form (Olaru, 2006), (Cloosterman, 2009)

For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

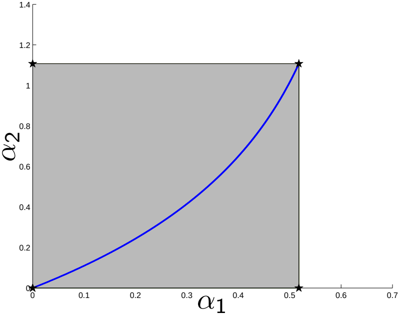
$$\Gamma(\rho) = \begin{pmatrix} \alpha_1(\rho) & 0 \\ 0 & \alpha_2(\rho) \end{pmatrix}$$

with

$$\alpha_i(\rho) = \int_0^\rho e^{\lambda_i s} ds$$

vertex = max or min $\alpha_i(\rho)$

$\lambda_1 = -1.5, \lambda_2 = 0.2$



$$\Gamma(\rho) = \sum_{j=0}^{2^n} \mu_j A_j$$

Polytopic embedding + gridding

For $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\Gamma(\rho) = \begin{pmatrix} \alpha_1(\rho) & 0 \\ 0 & \alpha_2(\rho) \end{pmatrix}$$

with

$$\alpha_i(\rho) = \int_0^\rho e^{\lambda_i s} ds$$

vertex = max or min $\alpha_i(\rho)$

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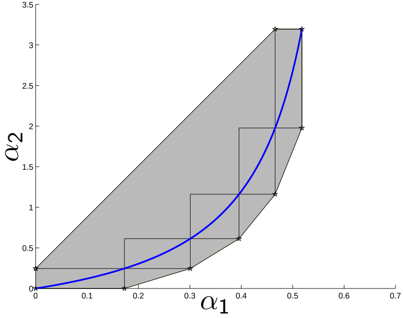
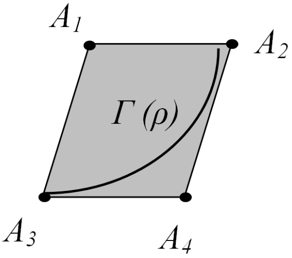


$$\Gamma(\rho) = \sum_{j=0}^{5\times 2^n} \mu_j A_j$$

Tractable LMI conditions



$$\sum_{j=1}^{N} \mu_j = 1,\ \mu_j > 0,\ \forall j = 1, \ldots, N,$$

$$\Gamma(\rho) = \sum_{i=1}^{N} \mu_j A_j,$$

- $\Lambda(\rho) = I + \Gamma(\rho)(A + BK) \in co\{I + A_j(A + BK), j = 1, \ldots, N\}$

$$P = P^T > 0$$

$$(I + A_j(A + BK))^T P (I + A_j(A + BK)) - P < 0,\ \forall j = 1, \ldots, N$$

- Finite number of LMI stability conditions !

Remarks about polytopic embeddings

- A polytopic representation with less vertex may be obtained based on Taylor series approximation

  (Hetel,Daafouz, Iung, TAC, 2006)
- The conservatism due to the use of an embedding may be tunned according to the desired numerical complexity
- Allow to use more efficient Lyapunov functions.

Lyapunov functions

Equivalent Difference inclusion

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y\ :\ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

with

$$\Lambda(h) = e^{hA} + \int_0^h e^{sA} ds BK$$

General Remarks on Linear Difference Inclusions (LDI) :

- Quadratic Lyapunov Functions (QLF) $V(x) = x^T P x$ are sufficient only for stability (not necessary)
- There are cases of LDI which are stable for which no QLF exists

(Dayawansa, Martin, IEEE TAC, 1999)

Lyapunov functions

Equivalent Difference inclusion

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y\ :\ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

General Remarks on Linear Difference Inclusions (LDI) :

- Necessary and Sufficient stability conditions : the existence of *Quasi-Quadratic* Lyapunov functions

$$V(x) = x^T \mathcal{L}_{[x]} x,$$

$$\mathcal{L}_{[x]} = \mathcal{L}_{[x]}^T = \mathcal{L}_{[ax]}, \quad \forall x \neq 0, a \in \mathbb{R}, a \neq 0$$

  s.t. the following relation is satisfied :

$$V(x) - \max_{h \in \mathcal{T}} V(\Lambda(h)x) > 0.$$

(Molchanov and Pyatniskii, SCL 1989)

Quasi-quadratic Lyapunov functions

For $V(x) = x^T \mathcal{L}_{[x]} x$ and polytopic LDI

- BMI criteria based on composite quadratic approximations

$$V_c(x) = \max_{i=1,\ldots,p} x^T L_i x, \;\; L_i = L_i^T > 0.$$

(Hu, Blanchini, Automatica, 2010)

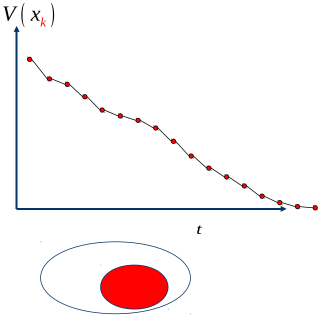
- level set of composite function $V_c(x)$ = intersection of ellipsoids
- no LMI existence criteria in the literature

Stability based on non-monotonous functions

(Megretzki, IEEE CDC 1994) ; (Krusezwski, Guerra, IEEE TAC 2008)

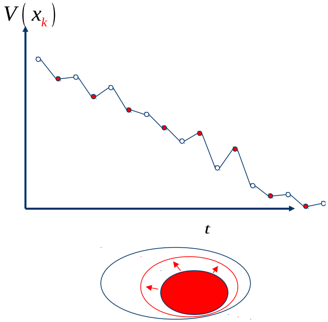
Classical approach

$$\forall x_k, \quad V(x_{k+1}) - V(x_k) < 0$$



New approach

$$\forall x_k, \quad V(x_{k+\alpha}) - V(x_k) < 0$$



Stability based on non-monotonous functions

Properties :

- $\alpha = 1$ - case of classical Lyapunov functions
- A LDI is stable iff there exist a finite $\alpha \in \mathbb{N}$ such as $V(x_{k+\alpha}) < V(x_k)$.

Stability based on non-monotonous functions

$$x^+ \in \mathcal{H}(x),\ \mathcal{H}(x) = \{y\ :\ y = \Lambda(h)x,\ h \in \mathcal{T} = [h_{min}, h_{max}]\},$$

Denote

- $\sigma = \{h^i\}_{i=0}^{\alpha-1}$ sequence of α sampling times and
- $\Phi_\sigma(\alpha)$ transition matrix associated to σ :

$$\Phi_\sigma(\alpha) = \begin{cases} \Lambda\left(h^{\alpha-1}\right)\ldots\Lambda\left(h^1\right)\Lambda\left(h^0\right), & \alpha > 0 \\ \mathbf{I}, & \alpha = 0. \end{cases}$$

with the function $V(x) = x^T P x$.

Proposition : *The equilibrium point $x = 0$ is asymptotically stable iff there exists a finite $\alpha \in \mathbb{N}^+$ s.t.*

$$\Phi_\sigma^T(\alpha) P \Phi_\sigma(\alpha) - P < 0$$

for all α lenght sequences with values in $\mathcal{T}$.

Stability based on non-monotonous functions

$\Lambda(h) \in co\mathcal{Z}$ where $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_N\}$
Consider the set of products of α matrices with values in $\mathcal{Z}$

$$\mathcal{Y}_\alpha(\mathcal{Z}) = \left\{ Y : Y = \Pi_{i=0}^{\alpha-1} Z_{\mu_i},\ Z_{\mu_i} \in \mathcal{Z} \right\}.$$

Proposition. *If there exist a positive integer* α *and a matrix* $P = P^T \succ 0$ *that satisfy*

$$P \succ Y^T P Y,\ \forall Y \in \mathcal{Y}_\alpha(\mathcal{Z}),$$

then the equilibrium point $x = 0$ *of the LDI is asymptotically stable.*

Stability based on non-monotonous functions

- finite number of LMI (complexity to be tunned according to number of vertex N and the horizon of analysis α)
- is there any relation with Quasi-Quadratic Lyapunov functions ?
- is there a constructive manner for obtaining a Quasi-Quadratic Lyapunov functions using P ?

Relation between the two approaches

Proposition. *If there exist a positive integer* α *and a matrix* $P = P^T \succ 0$ *that satisfy*

$$P \succ Y^T P Y, \ \forall Y \ \in \mathcal{Y}_\alpha(\mathcal{Z}),$$

then there exists a composite quadratic Lyapunov function for the LDI,

$$V_c(x) = \max_{i=1,\ldots,M} x^T L_i x \text{ s.t. } V_c(x) > \max_{\theta \in \mathcal{T}} V_c(\Lambda(\theta)x)$$

where $L_i, \ i = 1, \ldots, M = N^{\alpha-1}$ are an enumeration of the elements in the set

$$\Omega = \left\{ Q_\sigma^Z(\alpha) = \sum_{j=1}^{\alpha-1} \left(\Pi_{r=1}^j Z_{\mu_r}\right)^T P \left(\Pi_{r=1}^j Z_{\mu_r}\right) + P, \ \sigma = \{\mu_r\}_{r=1}^{N-1} \in \{1, \ldots, N\}^\alpha \right\}.$$

Numerical examples

$$A_c = \begin{pmatrix} -0.5 & 0 \\ 0 & 3.5 \end{pmatrix}, \quad B_c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } K = \begin{pmatrix} 1.02 & -5.62 \end{pmatrix}.$$

- $\Lambda(h)$ is Schur for any sampling interval $h \in [0, 0.46]$.
- $\Phi = (\Lambda(0.1))^6 \Lambda(0.43)$ is not Schur (exists a periodic unstable sequence).
- $h_k \in \{0.1, \ h_{max}\}$
- Exists a QLF for $h_{max} = 0.36$.
- with $\alpha = 7$, $h_{max} = 0.41$.

Numerical examples (Zhang,2001)

$$A_c = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} \text{ and } K = \begin{pmatrix} -3.75 & 11.5 \end{pmatrix}.$$

- $\Lambda(h)$ is Schur for any sampling interval $h \in [0, 0.172]$.
- $h_k \in [0.01, \; h_{max}]$
- for (Mirkin,2007), (Fridman,2004), (Hespanha,2008) $h_{max} < 1.36$
- with $\alpha = 1$, Taylor convex embedding, $h_{max} = 0.17$.

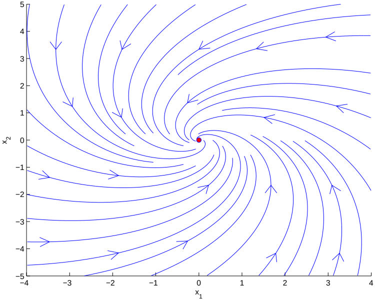
Numerical examples (Dayawansa, Martin, 1999)

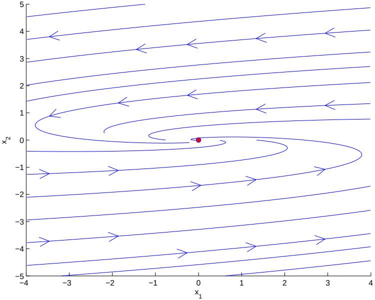
Sampled-data version of

$$\dot{x} = A_\sigma x, \ \ \sigma \in \{1, 2\}$$

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \ \ A_2 = \begin{pmatrix} -1 & -8 \\ 1/8 & -1 \end{pmatrix}$$

No QLP can be found





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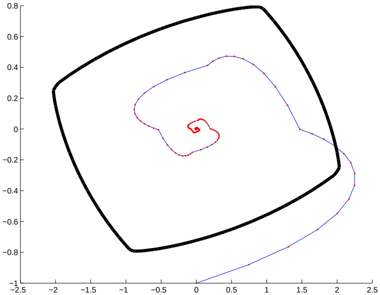
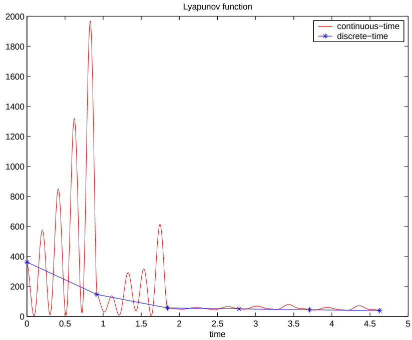


Positive Invariant Set based on Non-monotonous functions

Continuous-time approaches

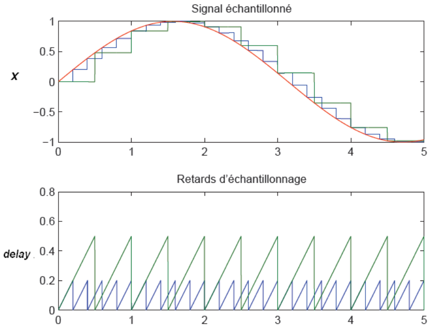
Main Discrete-time problem : inter-sampling behavior



Evolution of a Lyapunov function $V(x) = x^T Px$

- Strictly decreasing at $t = t_k$
  (sufficient condition for stability analysis)
- Increasing in between the sampling times
  (false evaluation of control performance)

Existing work - Continuous-time : time delay approach



$$u(t) = Kx(t_k) = Kx(t - \tau) \text{ with } \tau = t - t_k,\ 0 < \tau_k < h_{max}$$

Existing work - Continuous-time

- Fridman et al, 2004 (input delay approach)
- Mirkin, 2007 (robust control equivalent)
- Hespanha, 2008 (impulsive delay diff. eq.)

Advantage :

- Directly extend to performance study (decay rate)
- Take into account the inter-sampling behavior

Inconvenient :

- Do not take into account the sawtooth form of the delay (conservatism)

Note : discrete-time approaches implicitly take into account this aspect via the integration operator

Goal

Provide a continuous-time method that takes into account the sawtooth form of the delay
(advantage of discrete-time methods for conservatism reduction)

Case of quadratic Lyapunov functions

For $\dfrac{dx(t)}{dt} = Ax(t) + BKx(t - \tau(t)),\ \tau(t) := t - t_k,\ \forall t \in [t_k, t_{k+1})$

and $V(x) = x^T P x$

- Derivative of Lyapunov function

$$\frac{dV(x)}{dt} = 2x^T(t)P\left(Ax(t) + BKx(t-\tau)\right).$$

- Sawtooth evolution of delay can be introduced by using the integration operator $\Lambda(\cdot)$ used for the discrete-time model :

$$x(t) = \Lambda(t - t_k)\, x(t_k),\ \forall t \in [t_k, t_{k+1})$$

Case of quadratic Lyapunov functions

$$\frac{dV(x)}{dt} = 2x^T(t)P\left(Ax(t) + BKx(t-\tau)\right) < -2\alpha V(x(t)).$$

and

$$x(t) = \Lambda(\tau)\, x(t_k),\ \forall\tau \in [0, h_{max}]$$

Proposition : *If* $\exists\ P = P^T \succ 0\ G_1,\ G_2$ s.t.

$$\begin{pmatrix} A^TP + PA + G_1 + G_1^T + \alpha P & PBK - G_1\Lambda(\tau) + G_2^T \\ K^TB^TP - \Lambda^T(\tau)G_1^T + G_2 & -G_2\Lambda(\tau) - \Lambda^T(\tau)G_2^T \end{pmatrix} \prec \mathbf{0},$$

$\forall\tau \in [0, h_{max}]$ *then*

$$\frac{dV(x)}{dt} < -2\alpha V(x),\ \forall x \neq 0$$

Remark : satisfied if $\Lambda(\tau)$ is non-singular for all $\tau \in [0, h_{max}]$

Case of quadratic Lyapunov functions

Let $\Lambda(\tau) \in co\{A_j\}_{j=1}^{N}, \forall \tau \in [0, h_{max}]$.

Proposition : *If* $\exists\ P = P^T \succ 0\ G_1,\ G_2$ s.t.

$$\begin{pmatrix} A^T P + PA + G_1 + G_1^T + \alpha P & PBK - G_1 A_j + G_2^T \\ K^T B^T P - A_j^T G_1^T + G_2 & -G_2 A_j - A_j^T G_2^T \end{pmatrix} \prec \mathbf{0},$$

$$\forall l = 1, \ldots, N.$$

then

$$\frac{dV(x)}{dt} < -2\alpha V(x),\ \forall x \neq 0$$

Finite number of conditions may be obtained using the polytopic convex embedding for $\Lambda(\tau)$.

Non-singularity and Quasi-Quadratic Lyapunov functions

Remark : Existence of functions of the class $V_c(x)$ is necessary when Λ is non-singular

$$\frac{dx(t)}{dt} = Ax(t) + BKx(t_k), \forall t \in [t_k, t_{k+1}),$$

$$x(t_k) = \Lambda^{-1}(\tau)x(t)$$

$$\frac{dx}{dt} \in \mathcal{H}_c(x),\ \mathcal{H}_c(x) = \{\left(A + BK\Lambda^{-1}(\tau)\right) x, \tau \in [0, h_{max}]\},$$

based on (Molchanov, Pyatniski, 1989), (Hu, Blanchini, Automatica 2010)

Extension : Quasi-Quadratic Lyapunov functions

$$\frac{dx}{dt} \in \mathcal{G}_c(x,t),\ \mathcal{G}_c(x,t) = \{(A+BK)x, Ax + BKx(t_k)\},$$

For

$$V_c(x) = \max_{i=1,\dots,M} x^T L_i x$$

using

$$\max_{y(t)\in\mathcal{G}_c(x,t)} \nabla_{y(t)} V_c(x(t)) = 2\frac{dx}{dt}^T L_i x, \quad \text{for } x^T(L_j - L_i)x < 0$$

we obtain the following set of conditions

$$\begin{pmatrix} A^T L_i + L_i A + \lambda L_i - \sum_{i\neq j} \beta_{ij}(L_j - L_i) + G_1 + G_1^T & L_i BK - G_1\Lambda(\tau) + G_2^T \\ K^T B^T L_i - \Lambda^T(\tau) G_1^T + G_2 & -G_2\Lambda(\tau) - \Lambda^T(\tau)G_2^T \end{pmatrix} \prec \mathbf{0},$$

$$i,j = 1,\dots,M,\ \forall \tau \in [0, h_{max}].$$

Extension : Lyapunov - Razumikhin functions

- Consider the quadratic function $V(x) = x^T P x$, $P = P^T \succ 0$.
- Asymptotic stability conditions : $\dot{V}(x(t)) < 0$ whenever $V(x(t_k)) < \alpha V(x(t))$, with $\alpha > 1$
- Matrix Inequalities conditions : $P = P^T \succ 0$, a scalar $\epsilon > 0$, and matrices $G_1, G_2 \in \mathbb{R}^{n \times n}$ s.t.

$$\begin{pmatrix} A^T P + PA + \epsilon\alpha P + G_1 + G_1^T & PBK - G_1\Lambda(\theta) + G_2^T \\ K^T B^T P - \Lambda^T(\theta) G_1^T + G_2 & -G_2\Lambda(\theta) - \Lambda^T(\theta) G_2^T - \epsilon P \end{pmatrix} \prec \mathbf{0}.$$

$\forall \theta \in [0, \theta_{max}]$

Numerical example

Consider a continuous-time system described by :

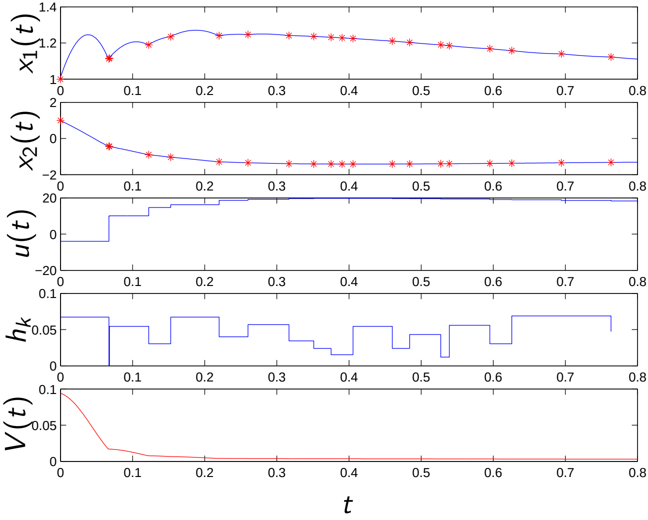
$$A = \begin{pmatrix} 1 & 15 \\ -15 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- $\lambda(A) = 1 \pm 15i$
- K - obtained by pole placement : $\lambda(A + BK) = -1 \pm i$

Stability analysis comparison :

- (Mirkin, TAC 2007) : $h \in [0, 0.014]$
- (Naghshtabrizi, Hespanha, Teel, SCL 2008) :$h \in [0, 0.033]$
- (Fujioka, Automatica 2009) : $h \in [0, 0.07]$
- continuous-time + polytopic embedding : $h \in [0, 0.09]$ (singularity for 0.092)
- Lyapunov-Razumikhn + polytopic embedding : $h \in [0, 0.14]$
- discrete-time approach : $h \in [0.01, 0.15]$

Numerical example



Conclusion and Perspective

- Robustness to sampling jitter
- Provide robust methods for stability
- Show how to reduce the conservatism of stability analysis by taking into account the sawtooth form of the delay
- Perspective : control design