EA 4353

Properties of NCGPC applied to nonlinear SISO systems with a relative degree one or two

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Outline

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  - Properties
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1. Relative degree of NL systems

- **System considered:**

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ y(t) = h(x(t)) \end{cases} \qquad (1)$$

- **Definition [Isidori 1995]:**

The nonlinear SISO system (1) is said to have a relative degree ρ around x^0 if:

(i) $L_g L_f^k h(x) = 0$ for all *x* in a neighbourhood of x^0 and all $k < \rho - 1$,

(ii) $L_g L_f^{\rho-1} h(x^0) \neq 0$

where $L_f h(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x) f_i(x)$

The relative degree ρ of (1) is said to be well-defined if (1) has the relative degree ρ at all points in an operating set [Chen 2001]

2. Unconstrained NCGPC

- **Criteria to minimize:**

$$J = \frac{1}{2}\int_0^T \left[\hat{e}(t+\tau)\right]^2 d\tau \qquad \text{where} \qquad \hat{e}(t+\tau) = \hat{y}(t+\tau) - \hat{\omega}(t+\tau)$$

- **The control law can be derived under the assumptions [Chen 2003]:**

1: zero dynamics exist and are asymptotically stable;

2: all states are accessible for measurements;

3: the system has a well-defined relative degree;

4: the output and the reference are sufficiently many times continuously differentiable with respect to time;

2. Unconstrained NCGPC

- **Taylor's series expansion:**

$$\hat{y}(t+\tau)=\sum_{k=0}^{\rho} y^{(k)}(t)\frac{\tau^k}{k!}+R(\tau^\rho) \quad \Rightarrow \quad \hat{y}(t+\tau)\approx\begin{bmatrix}1 & \tau & \cdots & \frac{\tau^\rho}{\rho!}\end{bmatrix}\begin{bmatrix}y(t)\\ \dot{y}(t)\\ \vdots \\ y^{(\rho)}(t)\end{bmatrix}$$

where
$$\begin{cases} y(t)=h(x(t)) \\ \dot{y}(t)=L_f h(x(t)) \\ \vdots \\ y^{(\rho)}(t)=L_f^\rho h(x(t))+L_g L_f^{\rho-1}h(x(t))u(x(t)) \end{cases}$$

In a similar way:

$$\hat{\omega}(t+\tau)\approx\begin{bmatrix}1 & \tau & \cdots & \frac{\tau^\rho}{\rho!}\end{bmatrix}\begin{bmatrix}\omega(t)\\ \dot{\omega}(t)\\ \vdots \\ \omega^{(\rho)}(t)\end{bmatrix}$$

2. Unconstrained NCGPC

- **Taylor's series expansion:**

$$\hat{y}(t+\tau) \approx \begin{bmatrix} 1 & \tau & \cdots & \dfrac{\tau^{\rho}}{\rho!} \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(\rho)}(t) \end{bmatrix}$$

$$\hat{\omega}(t+\tau) \approx \begin{bmatrix} 1 & \tau & \cdots & \dfrac{\tau^{\rho}}{\rho!} \end{bmatrix} \begin{bmatrix} \omega(t) \\ \dot{\omega}(t) \\ \vdots \\ \omega^{(\rho)}(t) \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{aligned} \Lambda(\tau) &= \begin{bmatrix} 1 & \tau & \cdots & \dfrac{\tau^{\rho}}{\rho!} \end{bmatrix} \\ Y(t) &= \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(\rho)}(t) \end{bmatrix} \\ \Omega(t) &= \begin{bmatrix} \omega(t) \\ \dot{\omega}(t) \\ \vdots \\ \omega^{(\rho)}(t) \end{bmatrix} \end{aligned} \right.$$

2. Unconstrained NCGPC

- **Criteria to minimize:**

From $E(t) = Y(t) - \Omega(t)$ ➡ $\hat{e}(t+\tau) = \Lambda(\tau)E(t)$

$$J = \frac{1}{2}\int_0^T [\hat{e}(t+\tau)]^2 d\tau \quad \Rightarrow \quad J = \frac{1}{2}E^t(t)\left[\int_0^T \Lambda^t(\tau)\Lambda(\tau)d\tau\right]E(t)$$

Let the prediction matrix $\Pi(T,\rho) = \int_0^T \Lambda^t(\tau)\Lambda(\tau)d\tau$

where $\Pi(T, \rho)$ is of dimensions $(\rho+ 1)\times(\rho+ 1)$

- **Criteria minimization:**

$$\left(\frac{\partial E(t)}{\partial u(t)}\right)^t \Pi(T,\rho)E(t) = 0$$

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2. Unconstrained NCGPC

- **Criteria minimization:**

$$E=\begin{bmatrix} y(t)-\omega(t) \\ \dot{y}(t)-\dot{\omega}(t) \\ \vdots \\ y^{(\rho)}(t)-\omega^{(\rho)}(t) \end{bmatrix} \Rightarrow E=\begin{bmatrix} h-\omega \\ L_f h-\dot{\omega} \\ \vdots \\ L_f^{\rho} h-\omega^{(\rho)} \end{bmatrix}+\begin{bmatrix} 0_{\rho\times 1} \\ uL_g L_f^{\rho-1}h \end{bmatrix} \Rightarrow \left(\frac{\partial E(t)}{\partial u(t)}\right)^t=\begin{bmatrix} 0_{1\times\rho} & L_g L_f^{\rho-1}h(x(t)) \end{bmatrix}$$

Let $D(x(t))=L_g L_f^{\rho-1}h(x(t))$

$$\left(\frac{\partial E(t)}{\partial u(t)}\right)^t \Pi(T,\rho)E(t)=0 \Rightarrow \begin{bmatrix} 0_{1\times\rho} & D \end{bmatrix}\Pi(T,\rho)\begin{bmatrix} h-\omega \\ \vdots \\ L_f^{\rho} h-\omega^{(\rho)}+Du \end{bmatrix}=0$$

$$\Rightarrow D\Pi_s\begin{bmatrix} h-\omega \\ \vdots \\ L_f^{\rho} h-\omega^{(\rho)}+Du \end{bmatrix}=0$$ where Π_s (dimensions $1\times(\rho+1)$) is the last row of $\Pi(T,\rho)$

2. Unconstrained NCGPC

- **Criteria minimization:**

$$D\Pi_s \begin{bmatrix} h-\omega \\ \vdots \\ L_f^\rho h - \omega^{(\rho)} + Du \end{bmatrix} = 0$$

The relative degree is supposed well-defined ➡ D cannot vanish for all $x \in X$: see (ii)

$$\Rightarrow \quad \Pi_s \begin{bmatrix} 0_{\rho\times 1} \\ Du \end{bmatrix} = \Pi_s \begin{bmatrix} \omega - h \\ \vdots \\ \omega^{(\rho)} - L_f^\rho h \end{bmatrix}$$

➡ $\Pi_{ss} Du = \Pi_s \begin{bmatrix} \omega - h \\ \vdots \\ \omega^{(\rho)} - L_f^\rho h \end{bmatrix}$ where Π_{ss} (dimensions 1×1) is the last element of vector Π_s

2. Unconstrained NCGPC

- **Resulting control law [Dabo 2009]:**

$$\Pi_{ss} D u = \Pi_s \begin{bmatrix} \omega - h \\ \vdots \\ \omega^{(\rho)} - L_f^{\rho} h \end{bmatrix} \quad \Rightarrow \quad u = D^{-1} \Pi_{ss}^{-1} \Pi_s \begin{bmatrix} \omega - h \\ \vdots \\ \omega^{(\rho)} - L_f^{\rho} h \end{bmatrix}$$

Let $K(T,\rho) = \Pi_{ss}^{-1} \Pi_s \quad \Rightarrow \quad K(T,\rho) = \begin{bmatrix} \frac{\rho!}{T^{\rho}} \frac{2\rho+1}{\rho+1} & \cdots & \frac{\rho!}{T^{\rho-1}} \frac{2\rho+1}{(l)!(\rho+l+1)} & \cdots & 1 \end{bmatrix}$

$$\Rightarrow \quad \boxed{u(x(t)) = \frac{-\sum_{l=0}^{\rho} K_{\rho l} \left[L_f^l h(x(t)) - \omega^{(l)}(t) \right]}{L_g L_f^{\rho-1} h(x(t))}}$$

where $K_{\rho l} = \frac{\rho!}{l!} \frac{2\rho+1}{(\rho+l+1) T^{\rho-1}}$ is the (l+1)th element of *K(T, ρ)*

2. Unconstrained NCGPC

- **Change of coordinates:**

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_\rho \end{bmatrix} = \begin{bmatrix} y-\omega \\ \dot{y}-\dot{\omega} \\ \vdots \\ y^{(\rho-1)}-\omega^{(\rho-1)} \end{bmatrix} \Rightarrow \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \vdots \\ \dot{z}_\rho = L_f^\rho h - \omega^{(\rho)} + uL_g L_f^{\rho-1} h \end{cases}$$

- **Resulting linear (and controllable) system:**

$$\begin{cases} \dot{Z} = AZ \\ O = CZ \end{cases} \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -K_{\rho 0} & -K_{\rho 1} & -K_{\rho 2} & \cdots & -K_{\rho(\rho-1)} \end{bmatrix} \quad \text{where} \quad K_{\rho l} = \frac{\rho!}{l!} \frac{2\rho+1}{(\rho+l+1)T^{\rho-1}}$$

Characteristic polynomial: $P_\rho(\lambda) = K_{\rho 0} + K_{\rho 1}\lambda + \cdots + \lambda^\rho = 0$

3. Case of relative degree ρ = 1

- **Characteristic polynomial:**

$$P_1(\lambda) = K_{10} + \lambda$$

- **Corresponding system:**

$$H_1(p) = \frac{G_1}{1 + \theta p}$$

- **Parameter identification:**

$$\begin{cases} K_{10} = \dfrac{1}{\theta} \\ K_{11} = 1 \end{cases} \quad \text{and} \quad K_{10} = \frac{\rho!}{T^{\rho}} \frac{2\rho + 1}{\rho + 1} \Rightarrow \boxed{\theta = 2T/3}$$

- **Theorem 1:**

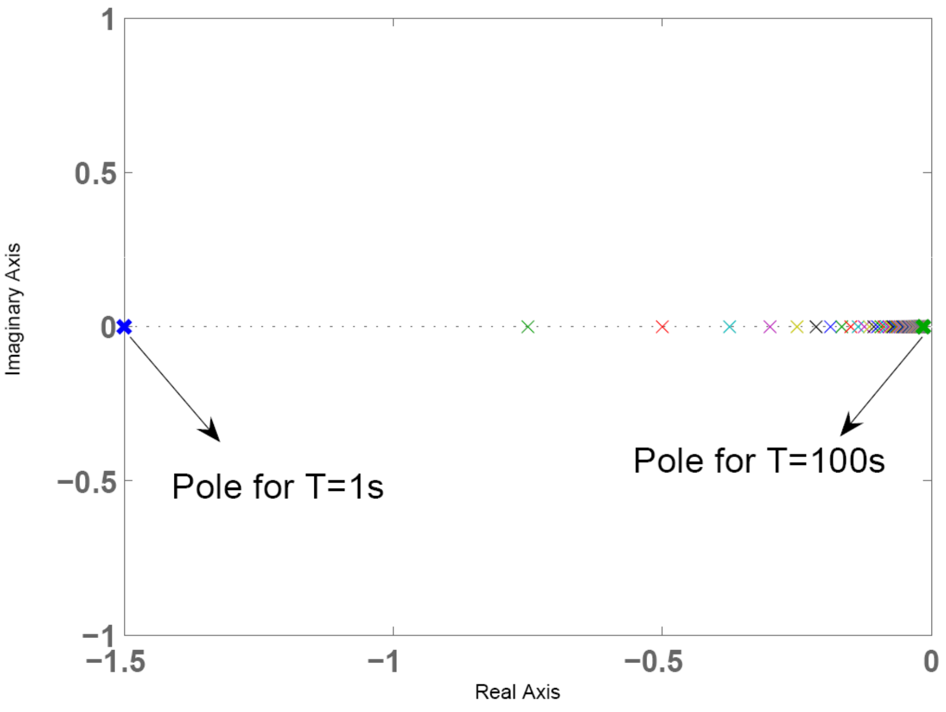
The application of NCGPC to SISO nonlinear system of dimension 1 equal to its relative degree, leads, in the right space of coordinates, to a linear 1st-order system with transfer function H_1 defined by a time constant θ and a static gain G_1 equal to the reference signal $\omega_1(t)$.

3. Case of relative degree ρ = 1: some properties

- **Characteristic parameters/times:**

Prediction horizon time	T
Time constant	$\theta = 2T/3$
Settling time at 5%	$t_{5\%} \approx 2T$
Cut-off frequency	$\omega_c = 3/2T$
Pole	$\lambda = -3/2T$

- **Closed-loop system stability:**



3. Case of relative degree ρ = 1: example

- **System considered:**

$$\begin{cases} \dot{x}(t) = 3x^2(t) + u(t) \\ y(t) = x(t) \end{cases}$$

- **System analysis:**
  - system dimension = 1
  - relative degree = 1

  ➡ No zero dynamics

- **Desired output:**
  - step

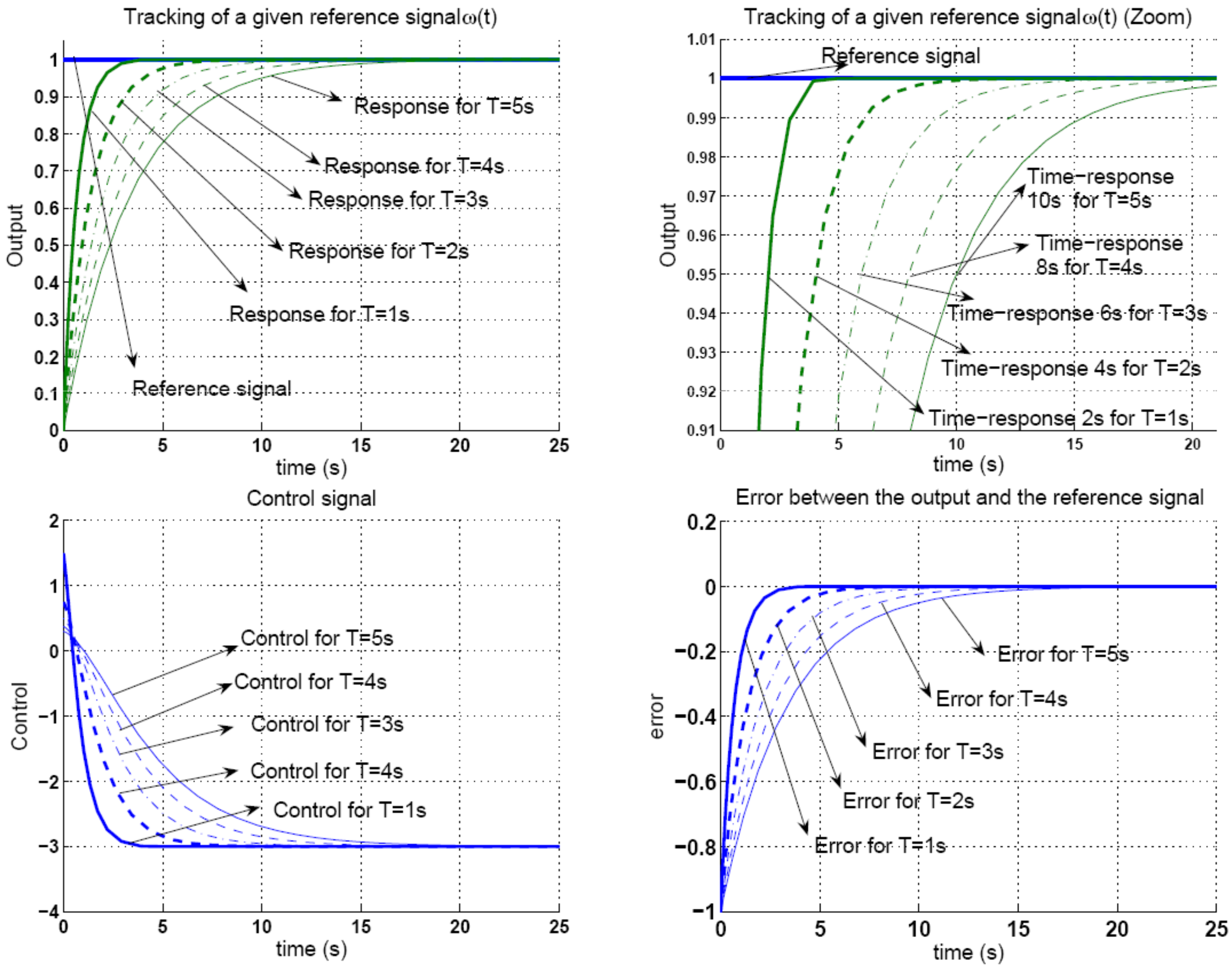
- **Parameter values vs. T:**

Prediction time T (s)	$K_1 = [K_{10}\ K_{11}]$
1	[1.5 1]
2	[0.75 1]
3	[0.5 1]
4	[0.375 1]
5	[0.3 1]

- **Control law:**

$$u(x(t)) = \frac{-\sum_{l=0}^{1} K_{1l}(T,1)\left[L_f^l h(x(t)) - \omega^{(l)}(t)\right]}{L_g L_f^0 h(x(t))}$$

3. Case of relative degree ρ = 1: simulation results



4. Case of relative degree ρ = 2

- **Characteristic polynomial:**

$$P_2(\lambda) = K_{20} + K_{21}\lambda + \lambda^2$$

- **Corresponding system:**

$$H_2(p) = \frac{G_2}{p^2 + 2\xi\omega_n p + \omega_n^2}$$

- **Parameters identification:**

$$\begin{cases} K_{20} = \omega_n^2 \\ K_{21} = 2\xi\omega_n \\ K_{22} = 1 \end{cases} \Rightarrow \begin{cases} \dfrac{\rho!}{T^\rho}\dfrac{2\rho+1}{\rho+1} = \omega_n^2 \\ \dfrac{\rho!}{T^{\rho-1}}\dfrac{2\rho+1}{\rho+2} = 2\xi\omega_n \end{cases} \Rightarrow \begin{cases} \omega_n = \sqrt{\dfrac{\rho!}{T^\rho}\dfrac{2\rho+1}{\rho+1}} \\ \xi(T,\rho) = \dfrac{1}{2}\dfrac{\rho!}{T^{\rho-2}}\sqrt{\dfrac{(\rho+1)(2\rho+1)}{(\rho+2)^2}} \end{cases} \Rightarrow \boxed{\begin{matrix} \omega_n \approx 1.83/T \\ \xi \approx 0.685 \end{matrix}}$$

- **Theorem 2:**

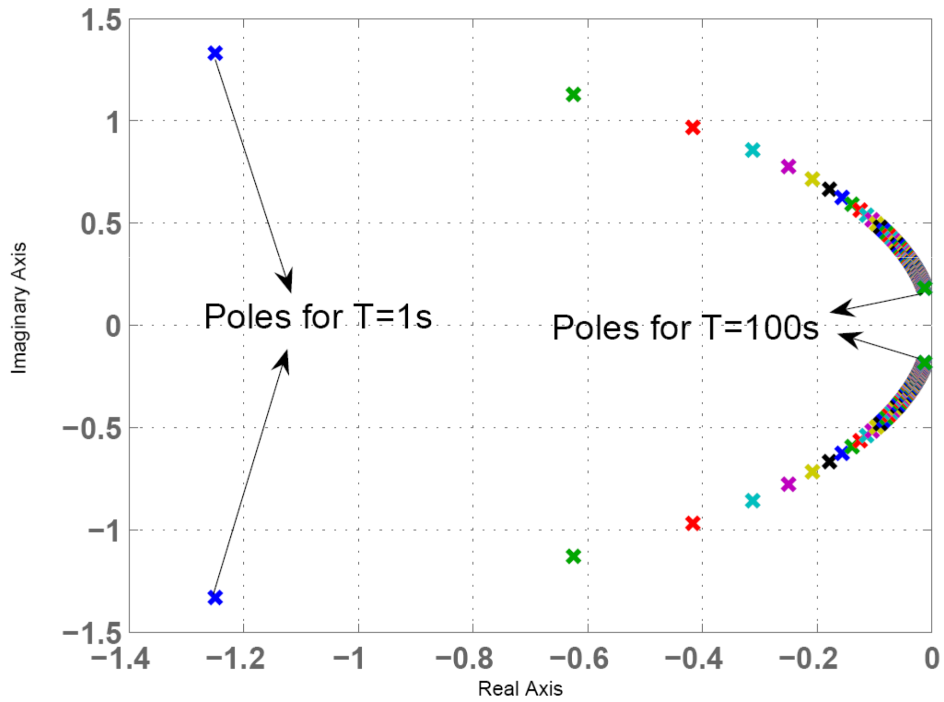
The application of NCGPC to SISO nonlinear system of dimension 2 equal to its relative degree, leads, in the right space of coordinates, to a 2nd-order linear transfer function with a constant damping ratio $\xi \approx 0.685$ and a natural frequency $\omega_n \approx 1.83/T$.

4. Case of relative degree ρ = 2: some properties

- **Characteristic parameters/times:**

Prediction horizon time	T
Rise time	$t_r \approx 1.47T$
Time-to-peak	$t_p \approx 2.34T$
Percent overshoot	$PO \approx 5.21$
Settling time at 5%	$t_{5\%} \approx 2.39T$
Poles	$\lambda_{1,2} = -\frac{1}{T}(1.25 \pm 1.33j)$
Resonant frequency	$\omega_r \approx 0.46/T$

- **Closed-loop system stability:**



4. Case of relative degree ρ = 2: example

- **System considered:**

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = 2x_1^2(t) - 3u(t) \\ y(t) = x_1(t) \end{cases}$$

- **System analysis:**
  - system dimension = 2
  - relative degree = 2
  - ➡ No zero dynamics
- **Desired output:**
  - step

- **Parameter values vs. T:**

Prediction time T (s)	K_2= [K_{20} K_{21} K_{22}]
1	[3.33 2.5 1]
2	[0.83 1.25 1]
3	[0.37 0.83 1]
4	[0.21 0.63 1]
5	[0.13 0.5 1]

- **Control law:**

$$u(x(t)) = \frac{-\sum_{l=0}^{2} K_{1l}(T,2)\left[L_f^l h(x(t)) - \omega^{(l)}(t)\right]}{L_g L_f h(x(t))}$$

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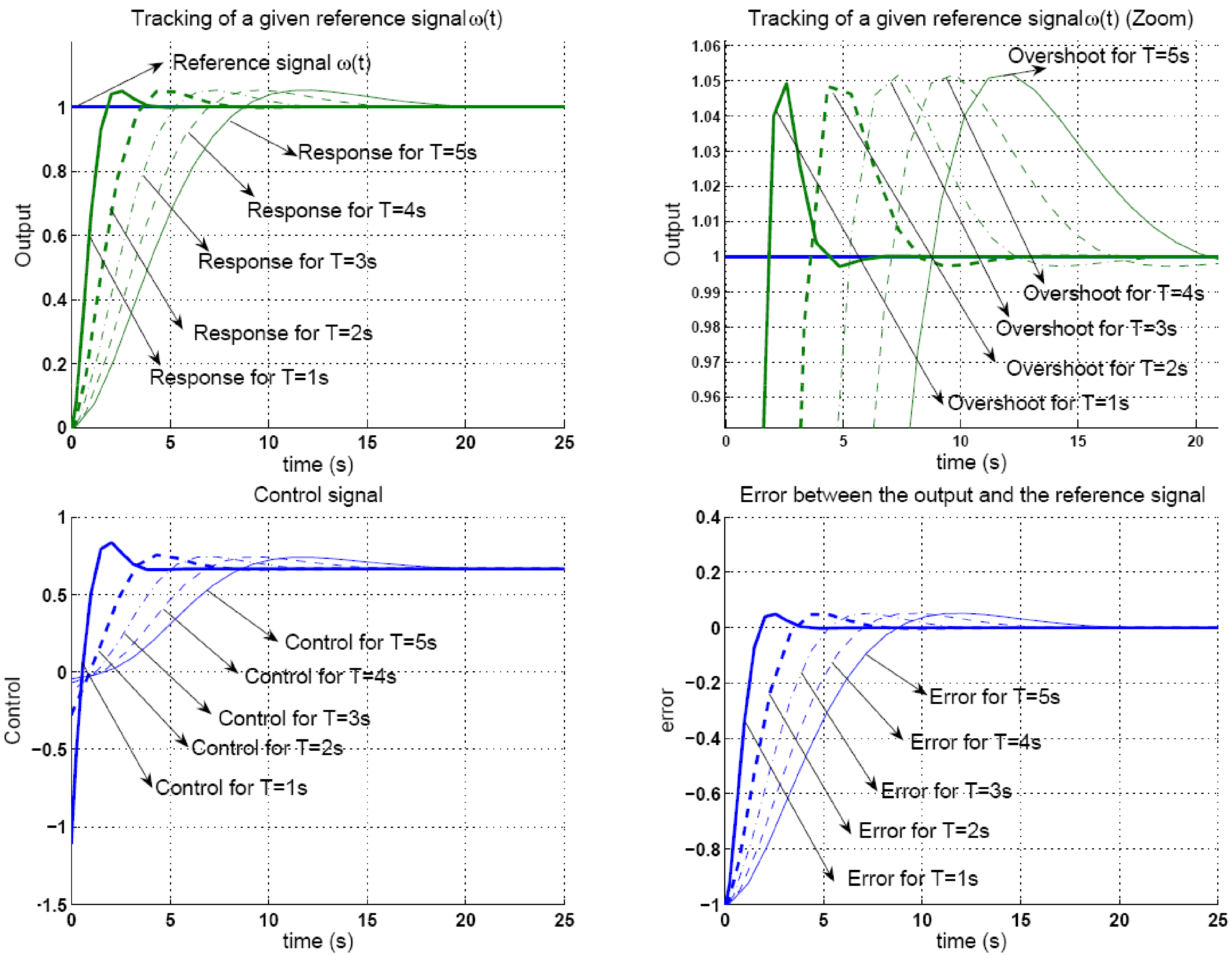
4. Case of relative degree ρ = 2: simulation results



Conclusion and future work

- Properties of NCGPC when applied to NL SISO systems with:
  - Relative degree equal to 1: stability, speed & accuracy (step)
  - Relative degree equal to 2: stability, speed & accuracy (step)

- Criteria based on error **and** control signal
- Robustness
- Nonlinear Discrete-time GPC
- Constraints on actuators
- Fault tolerant NDGPC

Thank you for your attention!

Appendix 1. Case of 1st-order system: some properties

	Function of T	Function of ω_c
Prediction horizon time	T	$T = \frac{3}{2\omega_c}$
Time constant	$\theta = \frac{2T}{3}$	$\theta = \frac{1}{\omega_c}$
Time response	$t_r = 2T$	$t_r = \frac{3}{\omega_c}$
Cutoff frequency	$\omega_c = \frac{3}{2T}$	ω_c
Pole	$\lambda = -\frac{3}{2T}$	$\lambda = -\omega_c$
Characteristic polynomial	$\lambda + \frac{3}{2T} = 0$	$\lambda + \omega_c = 0$

Appendix 2. Case of 2nd-order system: some properties

	Formula	Function of T	Function of ω_n
Prediction horizon time (T)	$T = \sqrt[\rho]{\frac{\rho!}{\omega_n^2}\frac{2\rho+1}{\rho+1}};\ T \in \mathbf{R}_*^+$	T	$T \simeq \frac{1.83}{\omega_n}$
Resonant frequency ω_r (case of $0 < \xi < \frac{\sqrt{2}}{2}$)	$\omega_r = \omega_n\sqrt{1-2\xi^2}$	$\omega_r \simeq \frac{0.46}{T}$	$\omega_r \simeq 0.25\omega_n$
Damped natural frequency (ω_d)	$\omega_d = \omega_n\sqrt{1-\xi^2}$	$\omega_d \simeq \frac{1.34}{T}$	$\omega_d \simeq 0.73\omega_n$
Undamped natural frequency (ω_n)	$\omega_n = \sqrt{\frac{\rho!}{T^\rho}\frac{2\rho+1}{\rho+1}}$	$\omega_n \simeq \frac{1.83}{T}$	ω_n
Rise time (t_r)	$t_r = \frac{(\pi-\arccos\xi)}{\omega_n\sqrt{1-\xi^2}}$	$t_r \simeq 1.47T$	$t_r \simeq \frac{3.19}{\omega_n}$
Time-to-peak (t_p)	$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$	$t_p \simeq 2.34T$	$t_p \simeq \frac{4.3}{\omega_n}$
Settling time at 5% ($t_{s_{5\%}}$)	$t_{s_{5\%}} = \frac{1}{\omega_n\xi}\ln\left(\frac{100}{5}\right) \simeq \frac{3}{\xi\omega_n}$	$t_{s_{5\%}} \simeq 2.39T$	$t_{s_{5\%}} \simeq \frac{4.38}{\omega_n}$
Settling time at 2% ($t_{s_{2\%}}$)	$t_{s_{2\%}} = \frac{1}{\omega_n\xi}\ln\left(\frac{100}{2}\right) \simeq \frac{4}{\xi\omega_n}$	$t_{s_{2\%}} \simeq 3.19T$	$t_{s_{2\%}} \simeq \frac{5.84}{\omega_n}$
Period of oscillation (T_p)	$T_p = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$	$T_p \simeq 4.71T$	$T_p \simeq \frac{8.62}{\omega_n}$
Characteristic polynomial	$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$	$\lambda^2 + \frac{5}{2T}\lambda + \frac{10}{3T^2} = 0$	$\lambda^2 + 1.37\omega_n\lambda + \omega_n^2 = 0$
Poles	$p_{1,2} = -\omega_n[\xi \pm j\sqrt{1-\xi^2}]$	$p_{1.2} \simeq -\frac{1}{T}(1.25 \pm 1.33j)$	$p_{1.2} \simeq -\omega_n(0.685 \pm 0.73j)$