Constrained control of uncertain discrete-time systems. An interpolation based approach

Hoai-Nam Nguyen, Sorin Olaru

E3S - Automatic Control Department {hoainam.nguyen, sorin.olaru}@supelec.fr

June 14, 2012

Project and collaborations

- PhD student: Hoai-Nam Nguyen
- Advisor: Sorin Olaru
- Project: A practical approach for constrained control of uncertain or time-varying discrete-time systems subject to bounded disturbances.
- Collaboration
  - Per-Olof Gutman: Faculty of Civil and Environmental Engineering Technion-Israel Institute of Technology, email: peo@technion.ac.il
  - Morten Hovd: Department of Engineering Cybernetics, Norwegian University of Science and Technology, email: morten.hovd@itk.ntnu.no

Outline

Outline

Abstract

Goal: Regulate to the origin

$$x(k+1) = A(k)x(k) + B(k)u(k) + D(k)w(k)$$

Constraints:

$$\begin{aligned} x(k) &\in X, X = \{x : F_x x \leq g_x\} \\ u(k) &\in U, U = \{u : F_u u \leq g_u\} \\ w(k) &\in W, W = \{w : F_w w \leq g_w\} \end{aligned}$$

Approach:

- Interpolation
- Minimizing an appropriate objective function
    1. Recursive feasibility
    2. Robustly asymptotically stable closed-loop behavior

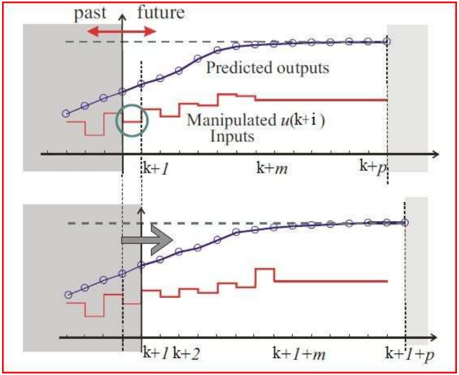
Outline

Constrained control - an overview

Many solutions, among them:

1. Optimal control
   - Almost open loop
2. Model predictive control
   - Implicit: optimal control problem solved at each sampling interval over a finite receding horizon,
   - Explicit: piecewise affine state feedback control laws computed off-line,
   - Extends with complexity to the uncertain plant case.
3. Vertex control law
   - Low cost alternative (from the complexity point of view)
   - Include the uncertain plant case
   - No optimality

Model predictive control - Implicit solution



1. Determine state $x(k)$.
2. Determine *on-line* optimal sequence of inputs over horizon.
3. Implement first input $u(k)$.
4. Wait for next sampling time $k = k + 1$.

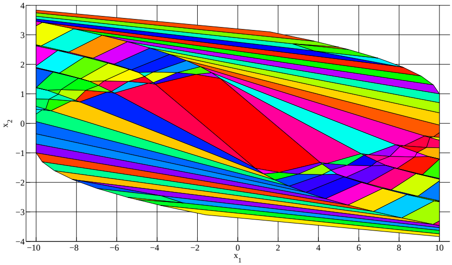
Model predictive control - Explicit solution

Feasible polyhedron C_N of initial states computed *off-line* by multi-parametric programming, where the parameterization is taken with respect to the initial state $x(0)$.

$$J = \sum_{i=1}^{14} \left(\| Qx(k+i) \|_2 + \| Ru(k+i-1) \|_2 \right)$$

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.3 \end{bmatrix} u(k)$$

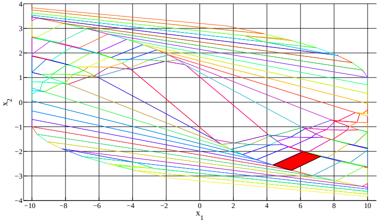
Constraints $-1 \le u \le 1$, $-10 \le x_1 \le 10$, $-5 \le x_2 \le 5$.

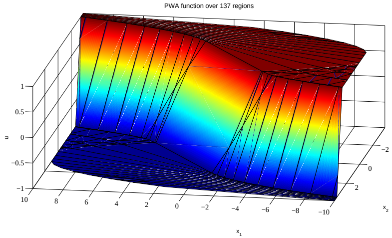


Model predictive control - Explicit solution, cont.

- C_N is divided in sub-polyhedra, an affine control law is found, **off-line**, for each sub-polyhedra, $u = K_i x + c_i$.
- The control is implemented **on-line** by finding to which sub-polyhedra the measured state belongs.

$$\begin{bmatrix} -0.19 & -0.98 \\ 0.19 & 0.98 \\ 0.31 & -0.95 \\ -0.29 & 0.96 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1.68 \\ -0.80 \\ 4.33 \\ -3.72 \end{bmatrix} \qquad u = -0.43x_1 - 2.23x_2 - 2.83$$



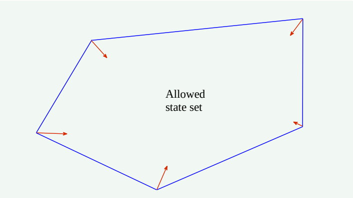


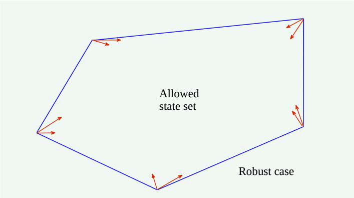
Model predictive control - Explicit solution, cont.

1) Advantages
    - Easy to implement
    - Fast on-line evaluation
2) Challenges
    - Number of controller regions can become large (for the double integrator example - the number of regions is 137)
    - Although off-line, computation time may become prohibitive

Vertex control

- The necessary and sufficient condition for constrained stabilization: at each vertex of the feasible set C_N exists a feasible control signal $u \in U$ that brings the state to $\text{int}(C_N)$ *in finite time*.
- See details in Gutman and Cwikel (1986).
- The extension to the uncertain plant case - Blanchini (1994).

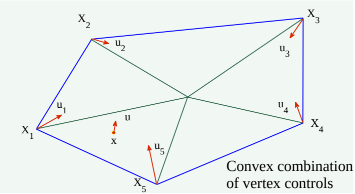


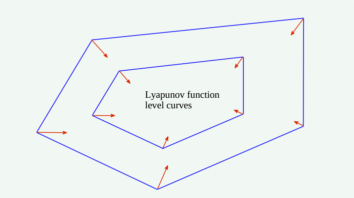


Vertex control - cont.

- The stabilizing controller is obtained by the convex combination of vertex controls in each sector
- Lyapunov level-functions are the scaling of the frontier of C_N.

$$x = \alpha X_1 + \beta X_5$$
$$u = \alpha u_1 + \beta u_5$$

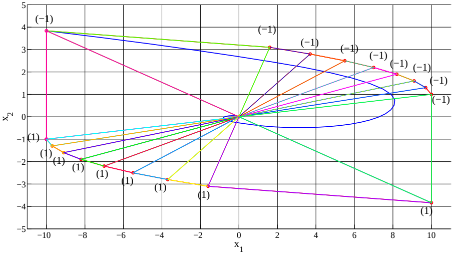


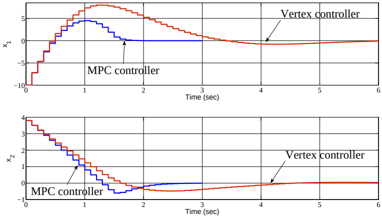


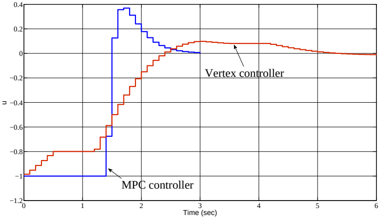
Vertex control - cont.

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.3 \end{bmatrix} u(k$$

$$-1 \leq u \leq 1,$$
$$-10 \leq x_1 \leq 10, \quad -5 \leq x_2 \leq 5$$
$$x(0) = [-10 \quad 3.84]^T$$







Outline

1. Abstract

2. Constrained control

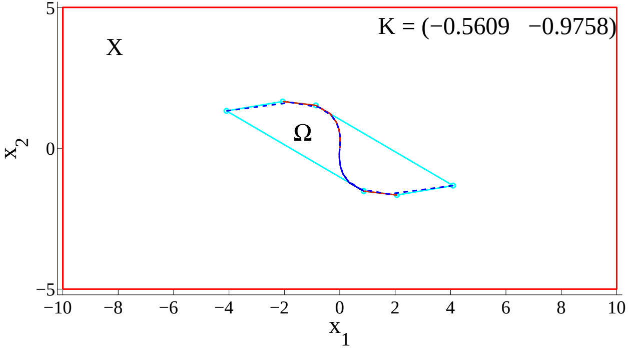
3. Interpolation based control
   - Set invariance
   - Interpolation based control via LP
   - Interpolation based control via QP
   - Interpolation via QP for uncertain systems with disturbances

4. Case study - Ball and plate

5. Conclusion

Maximal admissible set (MAS)

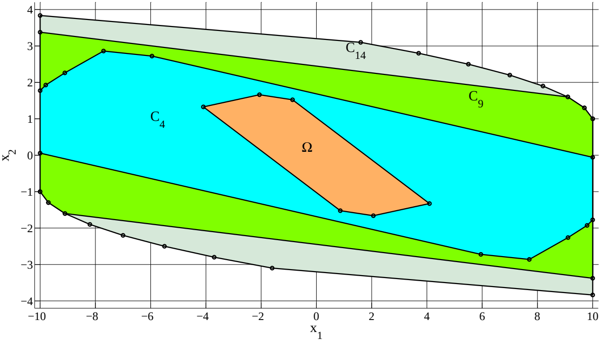
- For a given local feedback control $u = Kx$ there exists a polyhedral set $\Omega = \{x : F_o x \leq g_o\}$ such that for all $x(k) \in \Omega$, it follows $Kx(k) \in U$ and $x(k+1) = (A(k) + B(k)K)x(k) + D(k)w(k) \in \Omega$.
- See nominal LTI details in - Gilbert and Tan (1991) - Maximal admissible set (MAS)



Controlled invariant set

Using the one step pre-image set of the set $C_0 = \{x : F_0 x \leq g_0\}$ one can obtain the collection of states that can be brought in one step into C_0 by a feasible constrained control action.

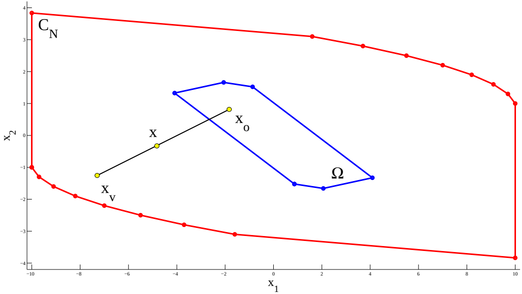
C_N is the set of states, that can be steered to the MAS Ω in no more than N step.



Interpolation based control - Implicit solution

$x \in C_N$ is decomposed as $x = cx_v + (1-c)x_o$ with $x_v \in C_N$, $x_o \in \Omega$

u_v: vertex control
u_o: local control



Theorem: The control law $u = cu_v + (1-c)u_o$ is feasible for all $x \in C_N$ and the closed loop system is robustly asymptotically stable.

Interpolation based control - Feasibility proof

Prove that $u(k) \in U$

$$\begin{aligned} F_u u(k) &= F_u(c u_v(k) + (1-c)u_o(k)) \\ &= cF_u u_v(k) + (1-c)F_u u_o(k) \\ &\le c g_u + (1-c)g_u = g_u \end{aligned}$$

And $x(k+1) \in C_N$

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ &= A(k)\{c x_v(k) + (1-c)x_o(k)\} + B(k)\{c u_v(k) + (1-c)u_o(k)\} \\ &= c\{A(k)x_v(k) + B(k)u_v(k)\} + (1-c)\{A x_o(k) + B u_o(k)\} \\ &\in C_N \end{aligned}$$

Interpolation based control - Stability proof

$\min c = c^*$ by finding *best* x_v, x_o.
Theorem: c^* is a positive and non-increasing Lyapunov function.

$$x(k) = c^*(k)x_v(k) + (1 - c^*(k))x_o(k)$$
$$u(k) = c^*(k)u_v(k) + (1 - c^*(k))u_o(k)$$

and

$$x(k+1) = A(k)x(k) + B(k)u(k) + D(k)w(k)$$
$$= c^*(k)x_v(k+1) + (1 - c^*(k))x_o(k+1)$$

Hence $c^*(k)$ is a feasible decomposition at time instant $k+1$.
By interpolation

$$x(k+1) = c^*(k+1)x_v^+(k+1) + (1 - c^*(k+1))x_o^+(k+1)$$

Hence $c^*(k+1) \leq c^*(k)$.

Nonlinear v.s. Linear Optimization

$$c^*(x) = \min_{c, x_o, x_v} c \text{ s.t. } \begin{cases} F_N x_v \leq g_N \\ F_o x_o \leq g_o \\ c x_v + (1 - c) x_o = x \\ 0 \leq c \leq 1 \end{cases}$$

Changing variables: $r_v = c x_v$, $r_o = (1 - c) x_o$.

$$c^*(x) = \min_{c, r_v, r_o} c \text{ s.t. } \begin{cases} F_N r_v \leq c g_N \\ F_o r_o \leq (1 - c) g_o \\ r_v + r_o = x \\ 0 \leq c \leq 1 \end{cases}$$

Interpolation based control via LP - Implicit solution

Computational scheme:

1. Measure state $x(k)$
2. Determine the optimal c^*
3. $u(k) = c^*(k)u_v(k) + (1 - c^*(k))u_o(k)$
4. Wait for next sampling time $k := k + 1$

At each time instant, only one LP with dimension $n_x + 1$ is solved.

System:

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$A(k) = \alpha(k)A_1 + (1 - \alpha(k))A_2$$
$$B(k) = \alpha(k)B_1 + (1 - \alpha(k))B_2$$

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

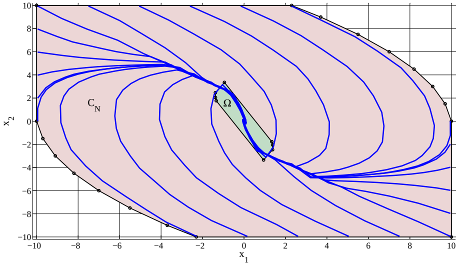
Interpolation based control via LP - Implicit solution, cont.

Constraints:

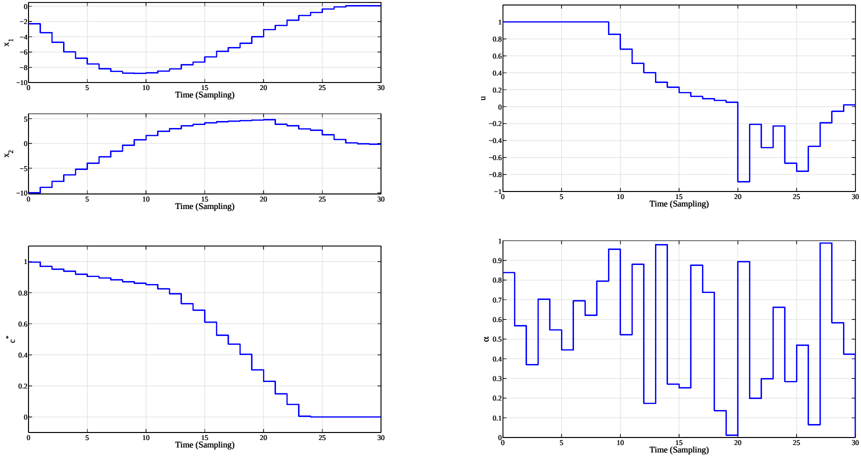
$$-10 \leq x_1 \leq 10, \ -10 \leq x_2 \leq 10,$$
$$-1 \leq u \leq 1$$

Local feedback controller:

$$K = [-1.8112 \quad -0.8092]$$



Interpolation based control via LP - Implicit solution, cont.



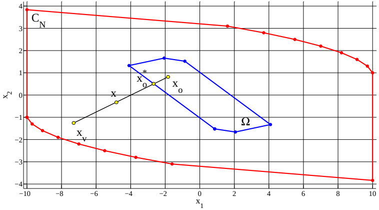
Geometric properties

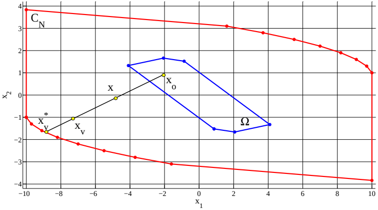
$$
\begin{aligned}
x &= cx_v + (1-c)x_o \\
&= c^* x_v + (1-c^*)x_o^*
\end{aligned}
$$

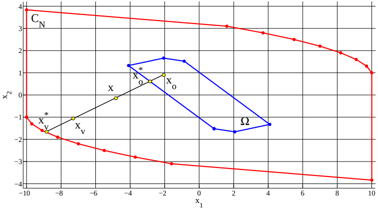
c is optimal iff $x_o \in \mathrm{Fr}(\Omega)$.

$$
\begin{aligned}
x &= cx_v + (1-c)x_o \\
&= c^* x_v^* + (1-c^*)x_o
\end{aligned}
$$

c is optimal iff $x_v \in \mathrm{Fr}(C_N)$







Geometric properties, cont.

- It is obvious that $c = 0$ for all $x \in \Omega$.
- For all $x \in C_N \setminus \Omega$, the smallest value c is reached if and only if $x_v \in \text{Fr}(C_N)$ and $x_o \in \text{Fr}(\Omega)$.

Fact: If $x \in C_N \setminus \Omega$ the smallest value c will be reached when the region $C_N \setminus \Omega$ is decomposed into polytopes. These polytopes can be further decomposed into simplices, each formed by s vertices of C_N and one vertex of Ω or one vertex of C_N and $n + 1 - s$ vertices of Ω with $1 \leq s \leq n$.

Geometric properties, cont.

Suppose x belongs to a simplex with $n+1$ vertices of $C_N \setminus \Omega$

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n+1} x_{n+1}$$

with $\alpha_i \geq 0$ and $\sum_{i=1}^{n+1} \alpha_i = 1$. Hence

$$\alpha = \begin{bmatrix} x_1 & x_2 & \ldots & x_{n+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

The corresponding control value

$$\begin{aligned} u &= \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_{n+1} u_{n+1} \\ &= \begin{bmatrix} u_1 & u_2 & \ldots & u_{n+1} \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \ldots & x_{n+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} \end{aligned}$$

The controller is an affine feedback state law whose gains are obtained simply by linear interpolation of the control values at the vertices of the simplex.

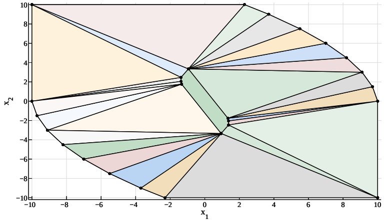
Explicit solution

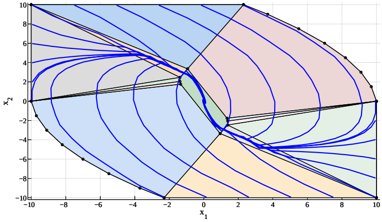
Algorithm

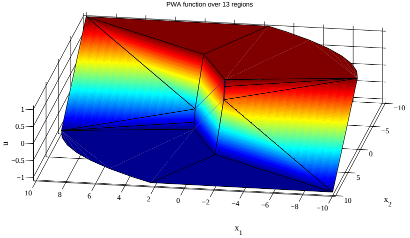
1. One gets the state space partition of $C_N \setminus \Omega$ by using explicit multi-parametric programming, where the parameter is x.
2. Decompose each partition of $C_N \setminus \Omega$ in a sequence of simplices.
3. In the MAS Ω the control law is $u = Kx$
4. In each simplex $C_k \subset C_N \setminus \Omega$ the control law is $u(x) = L_k x + v_k$ where

$$\begin{pmatrix} L_k & v_k \end{pmatrix} = \begin{bmatrix} u_1 & \dots & u_{n+1} \end{bmatrix} \begin{bmatrix} x_1^k & \dots & x_{n+1}^k \\ 1 & \dots & 1 \end{bmatrix}^{-1}$$

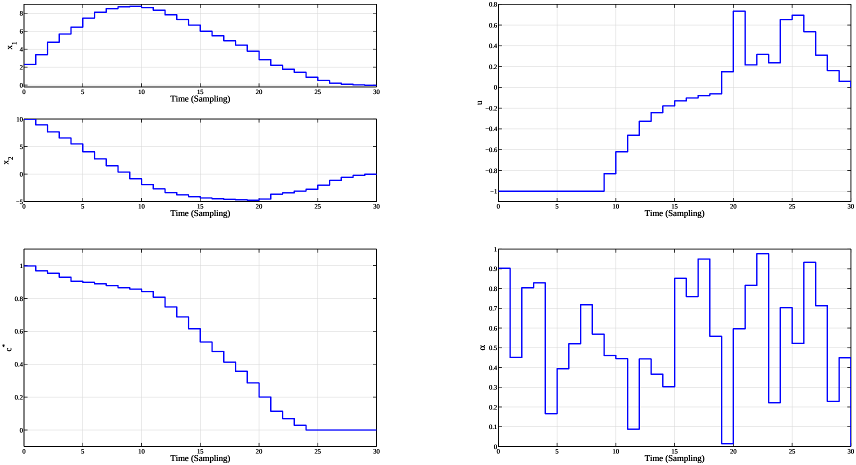
Explicit solution, cont.







Explicit solution, cont.



Problem formulation

For the system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

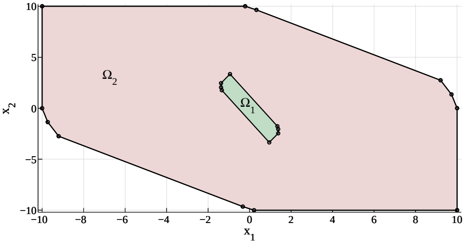
Given r invariant polyhedral sets with the feedback $u = K_i x$, $i = 1, 2, \ldots, r$.
The first polytope is used for the performance. The last $r - 1$ polytopes are used for enlarging the domain of attraction.

Question

- Is convex hull of the set of polytopes invariant?
- How to construct a control law for this region?

Interpolation via QP

$\Omega_i = \{x : F_o^{(i)} x \leq g_o^{(i)}\}$: robust positively invariant set for $u = K_i x$.



Ω = Convex hull(Ω_i)

$$x(k) = \lambda_1 \widehat{x}_1(k) + \lambda_2 \widehat{x}_2(k) + \ldots + \lambda_r \widehat{x}_r(k)$$

$\widehat{x}_i \in \Omega_i, \ \sum_{i=1}^{r} \lambda_i = 1.$

$$u(k) = \lambda_1 K_1 \widehat{x}_1(k) + \lambda_2 K_2 \widehat{x}_2(k) + \ldots + \lambda_r K_r \widehat{x}_r(k)$$

Interpolation via QP, cont.

Denote $x_i(k) = \lambda_i \widehat{x}_i(k)$. It follows that

$$F_o^{(i)} x_i \leq \lambda_i g_o^{(i)}$$

and

$$u(k) = K_1 x_1(k) + K_2 x_2(k) + \ldots + K_r x_r(k)$$

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) = A(k) \sum_{i=1}^{r} x_i(k) + B(k) \sum_{i=1}^{r} K_i x_i(k) \\ &= \sum_{i=1}^{r} x_i(k+1) \end{aligned}$$

with

$$x_i(k+1) = (A(k) + B(k)K_i) x_i(k)$$

Cost function determination

Q_i, R_i are weighting matrices

$$V_i(x_i) = x_i^T P_i x_i, i = 2, 3, \ldots, r$$

where P_i is chosen to satisfy

$$V_i(x_i(k+1)) - V_i(x_i(k)) \leq -x_i(k)^T Q_i x_i(k) - u_i(k)^T R u_i(k)$$

LMI condition

$$\begin{bmatrix} P_i - Q_i - K_i^T R_i K_i & (A + BK_i)^T P_i \\ P_i(A + BK_i) & P_i \end{bmatrix} \succeq 0$$

Interpolation via QP

Objective function

$$V = \sum_{i=2}^{r} x_i^T P_i x_i + \sum_{i=2}^{r} \lambda_i^2$$

Algorithm: at each time instant

$$\min\{V\}$$

subject to

$$\begin{cases} F_o^{(i)} x_i \le \lambda_i g_o^{(i)} \\ \sum_{i=1}^{r} x_i = x \\ \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \ge 0 \end{cases}$$

and apply as input the control action $u = \sum_{i=1}^{r} K_i x_i$.

Theorem: The interpolation based controller guarantees recursive feasibility and robust asymptotic stability for all initial states $x(0) \in \Omega$.

Feasibility and stability proof

Feasibility proof:

- Linearity of the system.
- Convexity of the feasible set.

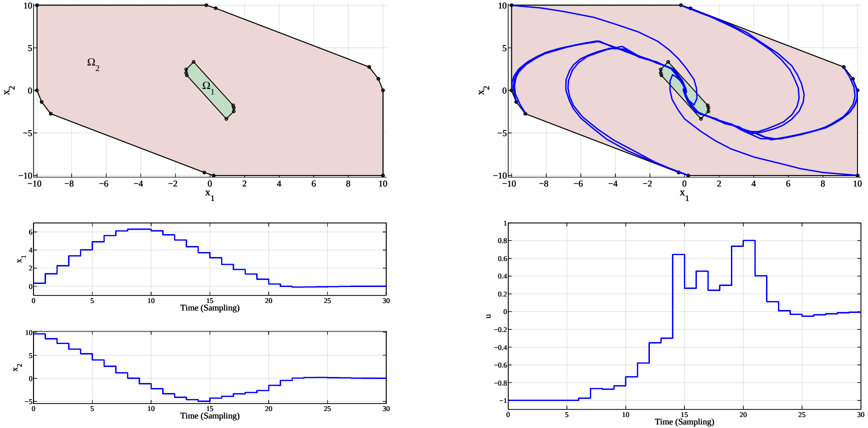
Stability proof:

- If $x_i^o(k)$ and $\lambda_i^o(k)$: optimal solution at time instant k. Then $x_i^o(k)$ and $\lambda_i^o(k)$: feasible solution at time instant $k+1$.
- $V(x)$ is a Lyapunov function

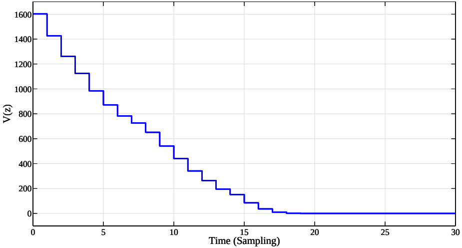
$$V(x(k+1)) - V(x(k)) \leq -\sum_{i=2}^{r}(x_i^T Q_i x_i + u_i^T R_i u_i)$$

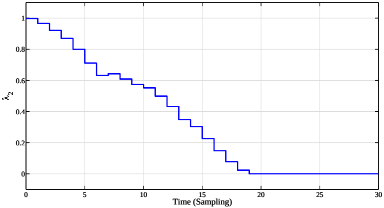
Interpolation via QP - Example

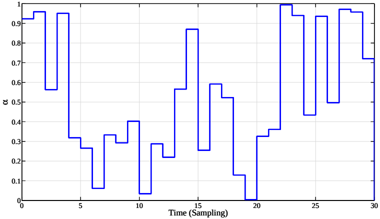
$$K_1 = [-1.8112 \quad -0.8092],$$
$$K_2 = [-0.0786 \quad -0.1010]$$



Interpolation via QP - Example, cont.







Interpolation via QP for uncertain systems with disturbances

System

$$x(k+1) = A(k)x(k) + B(k)u(k) + D(k)w(k)$$

Main ingredient

- A set of robust asymptotically stabilizing feedback controllers $u = K_i x,\ i = 1, 2, \dots, r$.
- A set of robust invariant polytopes Ω_i, $\Omega = \text{Convex hull}(\Omega_i)$

$$x(k) = \lambda_1 \widehat{x}_1 + \lambda_2 \widehat{x}_2 + \ldots + \lambda_r \widehat{x}_r$$

$\widehat{x}_i \in \Omega_i,\ \sum\limits_{i=1}^{r} \lambda_i = 1.$

$$u(k) = \lambda_1 K_1 \widehat{x}_1 + \lambda_2 K_2 \widehat{x}_2 + \ldots + \lambda_r K_r \widehat{x}_r$$

Preliminary definitions: Input to state stability

System

$$x(k+1) = A_c(k)x(k) + D(k)w(k)$$

ISS Stability: System is ISS iff

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \phi(\sup_{0 \leq i \leq k-1} \|w(i)\|)$$

ISS Lyapunov function: Function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function iff

$$\begin{cases} \gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|) \\ V(x(k+1)) - V(x(k)) \leq -\gamma_3(\|x(k)\|) + \theta(\|w(k)\|) \end{cases}$$

Theorem: System is input to state stable if it admits an ISS Lyapunov function.

Interpolation via QP for uncertain system with disturbances

$x_i(k) = \lambda_i \widehat{x}_i$. As in the previous slides

$$x(k) = \sum_{i=1}^{r} x_i(k+1)$$

where $z(k) = [x_1(k)^T \quad x_2(k)^T \quad \dots \quad x_2(k)^T]^T$ satisfies

$$z(k+1) = \Phi(k)z(k) + S(k)\eta(k)$$

$$\Phi(k) = \begin{bmatrix} A_c^{(1)} & 0 & \dots & 0 \\ 0 & A_c^{(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_c^{(r)} \end{bmatrix}, S(k) = \begin{bmatrix} D & 0 & \dots & 0 \\ 0 & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D \end{bmatrix}$$

$A_c^{(i)} = A(k) + B(k)K_i$

Interpolation via QP for uncertain system with disturbances

Q, R are weighting matrices.

$$V(z) = z^T P z$$

where

$$V(z(k+1)) - V(z(k)) \leq -x(k)^T Q x(k) - u(k)^T R u(k) + \tau \eta(k)^T \eta(k)$$

LMI condition

$$\begin{bmatrix} P - Q_1 - R_1 & 0 & \Phi^T P \\ 0 & \tau I & S^T P \\ P\Phi & PS & P \end{bmatrix} \succeq 0$$

$$\min \tau$$

Interpolation via QP for uncertain system with disturbances

At each time instant

$$\min\{V(z)\}$$

subject to

$$\begin{cases} F_o^{(i)} x_i \leq \lambda_i g_o^{(i)} \\ \sum_{i=1}^{r} x_i = x \\ \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \geq 0 \end{cases}$$

and apply as input the control action $u = \sum_{i=1}^{r} K_i x_i$.

Theorem: The interpolation based controller guarantees recursive feasibility and the closed loop system is ISS for all initial states $x(0) \in \Omega$.

Example

System

$$x(k+1) = A(k)x(k) + Bu(k) + w(k)$$

$$A(k) = \alpha(k)A_1 + (1-\alpha(k))A_2$$

$$A_1 = \begin{bmatrix} 1.1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.6 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

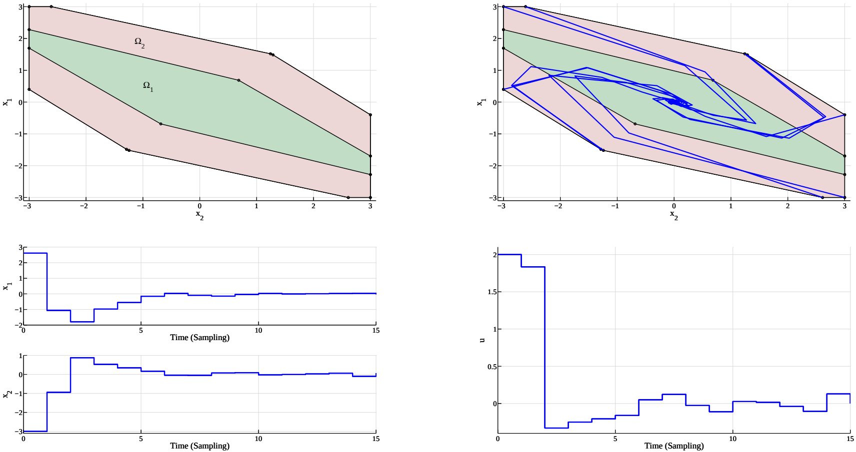
Constraints

$$\begin{array}{c} -3 \leq x_1 \leq 3, \quad -3 \leq x_2 \leq 3 \\ -2 \leq u \leq 2 \\ -0.1 \leq w_1 \leq 0.1, \quad -0.1 \leq w_2 \leq 0.1 \end{array}$$

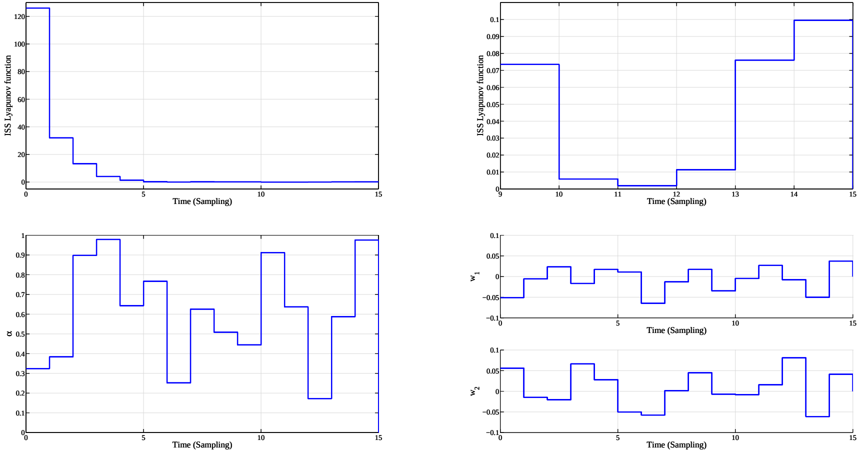
Controllers

$$\begin{array}{l} K_1 = [-0.8802 \quad -2.0355] \\ K_2 = [-0.3856 \quad -1.0024] \end{array}$$

Example, cont.



Example, cont.



Outline

System description

System description

- An mechanical plate.
- Two actuation mechanisms for tilting the plate around two orthogonal axes.
- A ball position sensor (touch screen)

System modeling

Only the model along the $x-$axis is considered here.

$$\begin{aligned} & x(k+1) - 1.6736x(k) + 0.3632x(k-1) + 0.2959x(k-2) = \\ & \quad = 0.0157u_x(k-1) + 0.0701u_x(k-2) \end{aligned}$$

Constraints

$$-1 \leq x \leq 1, \quad -1 \leq u_x \leq 1$$

State variables

$$\begin{cases} x_1(k) = x(k) \\ x_2(k) = x(k-1) \\ x_3(k) = x(k-2) \\ x_4(k) = u(k-1) \\ x_5(k) = u(k-2) \end{cases}$$

System modeling, cont.

State space model

$$x_n(k+1) = Ax_n(k) + Bu_x(k)$$

with

$$x_n = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$$

and

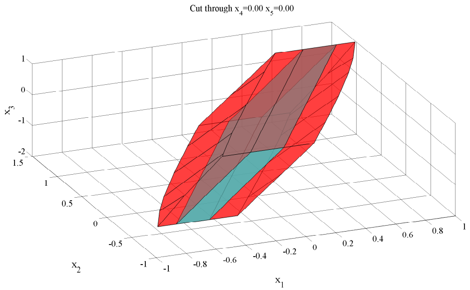
$$A = \begin{bmatrix} 1.6736 & -0.3632 & -0.2959 & 0.0157 & 0.0701 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

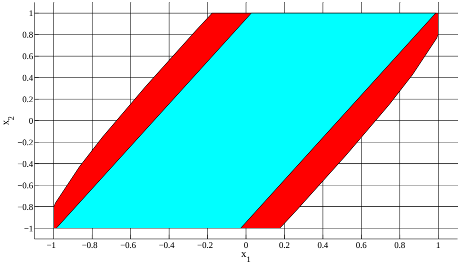
$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T$$

Local controller

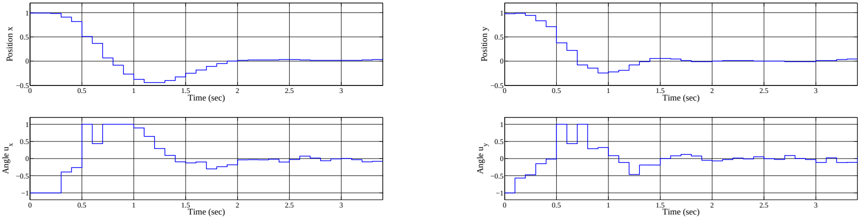
$$K = [-9.0103 \quad 4.5692 \quad 2.2384 \quad -0.5503 \quad -0.5303]$$

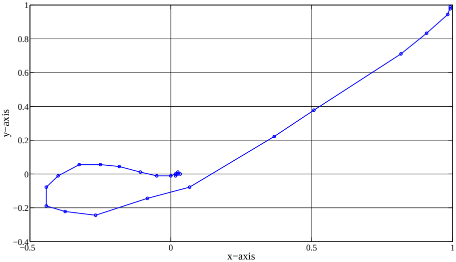
Ball and plate, cont.





Ball and plate, cont.





Outline

Conclusion

- Several novel interpolation schemes are introduced.
- A robust control law with guaranteed of constrained stability is obtained.
- The resulting control law is affine over a polyhedral partition of the state space.
- The proposed control law is considerably simpler with fewer polyhedral subsets in the explicit case, and with extremely simple and fast LP(or QP)-computations in the implicit case, and hence with less on-line computations than the MPC one.

Thank you for your attention!