Robust predictive control by zonotopic set-membership estimation

Commande prédictive robuste par des techniques d'observateurs à base d'ensembles zonotopiques

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Outline

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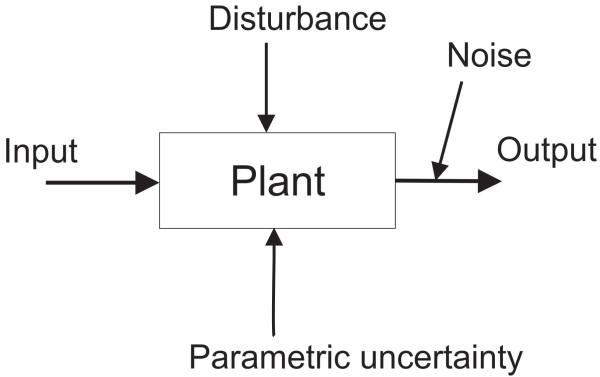
Context and motivation

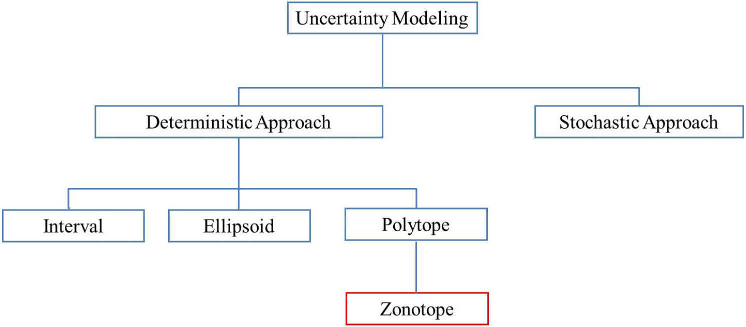
- Uncertain system



→ Taking into account these uncertainties to the system modeling.

Uncertainty modeling

- Two approaches for uncertainty modeling



Outline

Mathematical notations

- **Interval**: $[a; b] = \{x : a \leq x \leq b\}$
  Unitary interval: $\mathbf{B} = [\text{-}1;1]$
  Matrix interval: $[A]$ with A_{ij} intervals.
- **Minkowski sum**: $X \oplus Y = \{x + y : x \in X, y \in Y\}$.
- **Zonotope**: a convex symmetric polytope
  m-zonotope: the set $p \oplus H\mathbf{B}^m = \{p + Hz : z \in \mathbf{B}^m\}$, with a vector $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n \times m}$.
- P-**radius** of a zonotope $Z = p \oplus H\mathbf{B}^m$:
  $L = \max(\|z - p\|_P^2)$, with $z \in Z$ and $P = P^T \succ 0$.

Mathematical notations

Example: $p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $H_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$,

$p_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 0.4 & 3 \\ 3 & 0.2 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$L_1 = \max_{z \in Z_1} \|z\|_P^2 = 72$, $L_2 = \max_{z \in Z_2} \|x\|_P^2 = 37$.

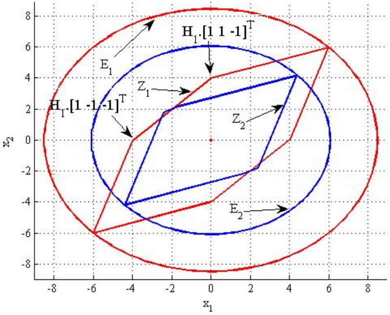


Figure: Zonotopes and ellipsoids representing the associated P-radius

Outline

Problem formulation

- Linear discrete-time system:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_k \\ y_k = Cx_k + v_k \end{cases} \quad (1)$$

where

- $x_k \in \mathbb{R}^{n_x}$ system state vector,
- $y_k \in \mathbb{R}^{n_y}$ measured output vector,
- $\omega_k \in \mathbb{R}^{n_x}$ state disturbances,
- $v_k \in \mathbb{R}^{n_y}$ measurement noise.
- Assumptions
  1. Detectable, stabilizable.
  2. $\omega_k \in W, v_k \in V$, with W a zonotope, V a box (for simplicity, W, V can be centered in the origin).
  3. x_0 unknown and $x_0 \in X_0$, with X_0 is a zonotope.

Goal: **Estimate** the system state under uncertainties and **Stabilize** system (1).

Outline

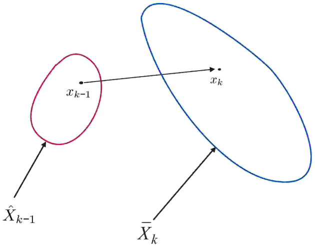
Guaranteed state estimation algorithm

Consider the autonomous system:

$$\begin{cases} x_{k+1} = Ax_k + \omega_k \\ y_k = Cx_k + v_k \end{cases} \quad (2)$$

General algorithm for set-membership estimation:

- Step 1: (*Prediction*) Compute a set $\bar{X}_k$ that offers a bound for the uncertain trajectory of the system: $\bar{X}_k = A\hat{X}_{k-1} \oplus W$



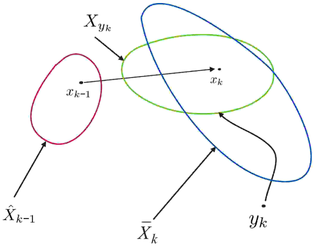
Guaranteed state estimation algorithm

Consider the autonomous system:

$$\begin{cases} x_{k+1} = Ax_k + \omega_k \\ y_k = Cx_k + v_k \end{cases} \quad (3)$$

General algorithm for set-membership estimation:

- Step 1: (*Prediction*)
- Step 2: (*Measurement*) Compute the measurement consistent state set X_{y_k} by using the measurement y_k.



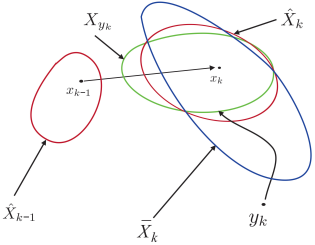
Guaranteed state estimation algorithm

Consider the autonomous system:

$$\begin{cases} x_{k+1} = Ax_k + \omega_k \\ y_k = Cx_k + v_k \end{cases} \tag{4}$$

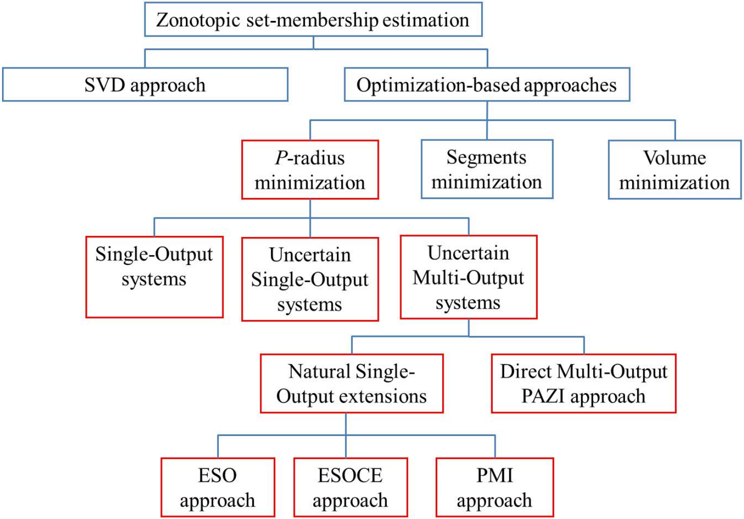
General algorithm for set-membership estimation:

- Step 1: (*Prediction*)
- Step 2: (*Measurement*)
- Step 3: (*Correction*) Compute an outer approximation $\hat{X}_k$ of the intersection between X_{y_k} and $\bar{X}_k$.



→ Similar to Kalman filter.

Guaranteed state estimation by zonotopes



Single-Output case

- Given the system:

$$\begin{cases} x_{k+1} = Ax_k + \omega_k \\ y_k = c^T x_k + v_k \end{cases} \qquad (5)$$

with $\omega_k \in W$ a zonotope, $v_k \in V = \sigma \mathbf{B}^1$ an interval.

At the time instant k:
- Guaranteed state set at $k-1$: $\hat{X}_{k-1} = p \oplus H\mathbf{B}^r$.
- Measured output: y_k.
- Rewrite $W = F\mathbf{B}^{n_x}$.
  - Prediction:

$$\bar{X}_k = Ap \oplus \begin{bmatrix} AH & F \end{bmatrix} \mathbf{B}^{r+n_x} \qquad (6)$$

  - Measurement: $X_{y_k} = \{x \in \mathbb{R}^n : |c^T x - y_k| \leq \sigma\}$
  - Correction: State estimation $\hat{X}_k$ the outer approximation of the intersection between a zonotope and a strip.

Intersection of a zonotope and a strip

Property 1: Zonotopic outer approximation of the intersection between a zonotope and a strip[1]

Given:

- zonotope $Z = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$,
- strip $S = \{x \in \mathbb{R}^n : |c^T x - d| \leq \sigma\}$,
- vector $\lambda \in \mathbb{R}^n$.

Define:

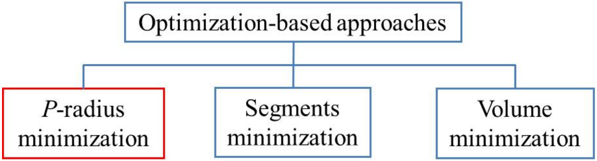
- vector $\hat{p}(\lambda) = p + \lambda(d - c^T p) \in \mathbb{R}^n$,
- matrix $\hat{H}(\lambda) = [(I - \lambda c^T)H \ \sigma\lambda] \in \mathbb{R}^{n\times(m+1)}$.

Then the following expression holds:
$Z \cap S \subseteq \hat{Z}(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)B^{r+1}$.

[1] Alamo et al. (2005)

Single-Output case

Using Property 1: $\hat{X}_k(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+n_x+1}$,
with $\hat{p}(\lambda) = Ap + \lambda(y_k - c^T Ap)$
and $\hat{H}(\lambda) = \left[(I - \lambda c^T)\left[AH \quad F\right] \quad \sigma\lambda\right]$
How to choose λ?



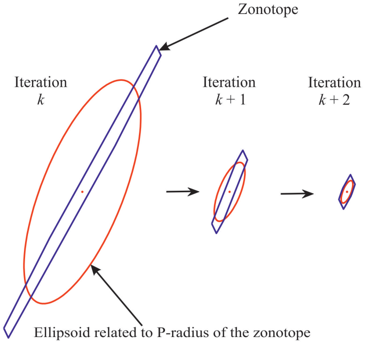
- Minimizing the segments of the zonotope: simple but not efficient.
- Minimizing the volume of the zonotope: more accurate and more complex.

$\rightarrow$ The proposed P-radius based approach combines the advantages of the two existing approaches.

Guaranteed state estimation by zonotopes

New criterion [2] to compute λ

Compute a matrix $P = P^T \succ 0$ and a vector λ such that at each sample time, the P-radius of the zonotopic state estimation set is non-increasing.



[2] Le et al. (2011)

Guaranteed state estimation by zonotopes

- The non-increasing condition leads to:

$$\max_{\hat{z}\in\mathbf{B}^{r+n_x+1}} \|\hat{H}\hat{z}\|_P^2 \leq \max_{z\in\mathbf{B}^r} \beta\|Hz\|_P^2 + \max_{s\in\mathbf{B}^{n_x}} \|Fs\|_2^2 + \sigma^2 \quad (7)$$

with $\hat{z} = \begin{bmatrix} z^T & s^T & \eta \end{bmatrix}^T$, $\eta \in \mathbf{B}^1$, $\beta \in (0;1)$.

- Denote by: $L_k = \max_{x\in\hat{X}_k}(\|x - p_k\|_P^2)$ and $\epsilon = \max_{s\in\mathbf{B}^{n_x}} \|Fs\|_2^2$
- Then (7)$\leftrightarrow L_{k+1} \leq \beta L_k + \epsilon + \sigma^2$
- Equivalent BMI (Bilinear Matrix Inequality) problem:

$$\begin{bmatrix} \beta P & 0 & 0 & A^T P - A^T c Y^T \\ * & F^T F & 0 & F^T P - F^T c Y^T \\ * & * & \sigma^2 & Y^T \sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \quad (8)$$

with β, P and $Y = P\lambda$ as decision variables.

Minimization of the P-radius

- At infinity: $L_\infty = \beta L_\infty + \epsilon + \sigma^2 \Leftrightarrow L_\infty = \frac{\epsilon+\sigma^2}{1-\beta}$
- Consider an ellipsoid: $E = \{x : x^T P x \leq \frac{\epsilon+\sigma^2}{1-\beta}\} \Leftrightarrow E = \{x : x^T \frac{(1-\beta)P}{\epsilon+\sigma^2} x \leq 1\}$
- To minimize the size of the guaranteed set, the ellipsoid of the smallest diameter must be found $\Rightarrow$ Eigenvalue Problem (EVP):[3]

$$\max_{\tau,\beta,P} \tau$$

subject to BMI

$$\begin{cases} \tau > 0 \\ P = P^T \succ 0 \\ \frac{(1-\beta)P}{\epsilon+\sigma^2} \succeq \tau I \end{cases} \tag{9}$$

Diameter of the obtained ellipsoid: $\frac{2}{\sqrt{\tau^*}}$.

[3] Boyd et al. (1994)

Guaranteed state estimation by zonotopes

Global optimization problem

Solve $\max_{\tau,\beta,P,Y} \tau$
subject to BMIs

$$\begin{cases} \tau > 0 \\ \frac{(1-\beta)P}{\epsilon+\sigma^2} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & A^T P - A^T c Y^T \\ * & F^T F & 0 & F^T P - F^T c Y^T \\ * & * & \sigma^2 & Y^T \sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \end{cases} \quad (10)$$

-Solution obtained off-line by the Penbmi solver [4] or a search loop on $\beta \in (0, 1)$.

-Detectability leads to a feasible solution.

[4] Kočvara and Stingl (2003)

Extension to Single-Output systems with interval uncertainties [5]

A is unknown, Schur stable, $A \in [A]$.

Global optimization problem

Solve $\max_{\tau,\beta,P,Y} \tau$

subject to BMIs

$$\begin{cases} \tau > 0 \\ \frac{(1-\beta)P}{\epsilon+\sigma^2} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & S_i^T P - S_i^T c Y^T \\ * & F^T F & 0 & F^T P - F^T c Y^T \\ * & * & \sigma^2 & Y^T \sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \end{cases} \tag{11}$$

with $\beta \in (0,1)$ and S_i the vertices of $[A]$, $i = 1, \ldots, 2^q$ and q the number of interval elements in $[A]$.

[5] Le et al. (2012a)

Example

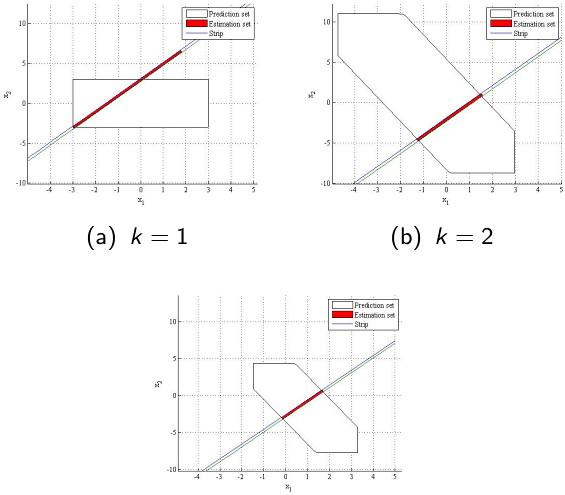
Consider the following linear discrete-time invariant system:

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+0.3\delta \end{bmatrix} x_k + 0.02 \begin{bmatrix} -6 \\ 1 \end{bmatrix} \omega_k \\ \\ y_k = \begin{bmatrix} -2 & 1 \end{bmatrix} x_k + 0.2 v_k \end{cases} \tag{12}$$

with $\|v_k\|_\infty \leq 1, \|\omega_k\|_\infty \leq 1$ and $|\delta| \leq 1$.

The initial state belongs to the box $3\mathbf{B}^2$.

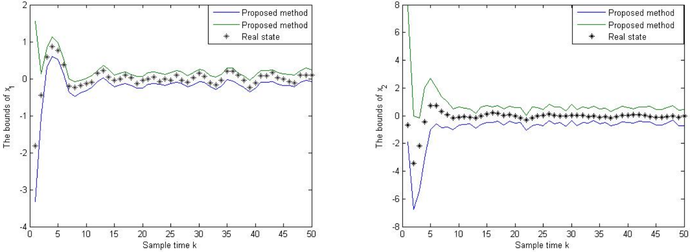
Simulation results



(a) $k = 1$ (b) $k = 2$ (c) $k = 3$

Figure: Evolution of the guaranteed state estimation $\hat{X}_k$

Simulation results

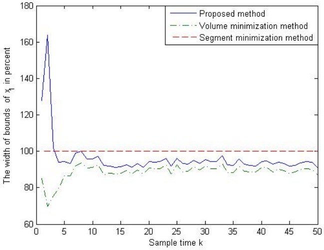


(a) x_1 (b) x_2

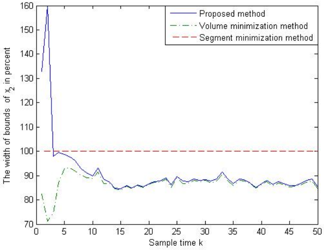
Figure: Guaranteed bounds obtained by proposed method

-The real state is found inside the guaranteed bound
$\rightarrow$ good estimation.

Simulation results



(a) x_1



(b) x_2

Figure: Comparison of the bound's width obtained by different methods in percent

Simulation results

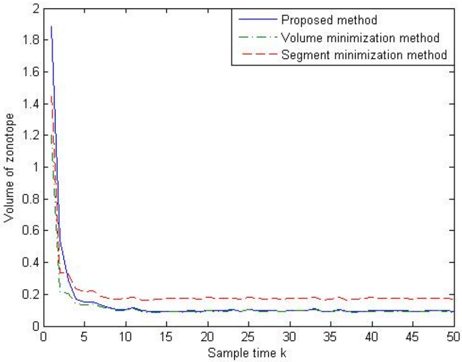


Figure: Comparison of the volume of zonotopic state estimation set obtained by different methods

- The P-radius based approach is better than the segment minimization approach and similar to the volume minimization approach.

Simulation results

Table: Total computation time after 50 time instants

Approach	Time(second)
Segment minimization	0.0312
P-radius minimization (without off-line optimization (11) included)	0.0312
P-radius minimization (with off-line optimization (11) included)	0.9828
Volume minimization	10.3273

- Conclusion: The P-radius based approach offers a trade-off between the complexity and the precision.

Multi-Output case

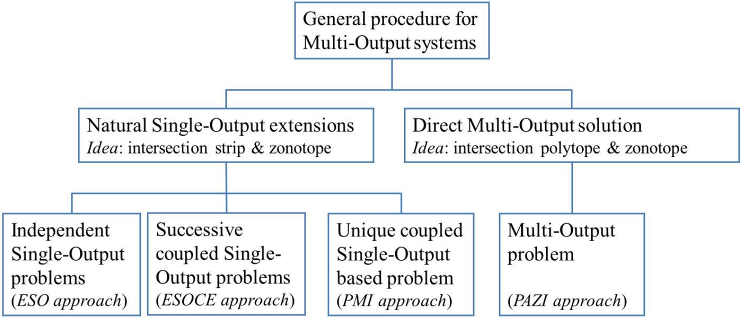
- Given a system

$$\begin{cases} x_{k+1} = Ax_k + \omega_k \\ y_k = Cx_k + v_k \end{cases} \tag{13}$$

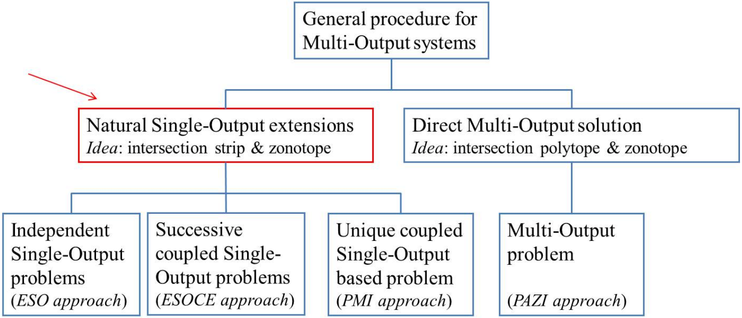
with

- $\omega_k \in W$ a zonotope $\subset \mathbb{R}^{n_x}$
- $v_k \in V$ a box $\subset \mathbb{R}^{n_y}$, $V = \Sigma \mathbf{B}^{n_y}$ with $\Sigma = diag(\sigma_1, \ldots, \sigma_{n_y})$

Solution for Multi-Output case



Natural extensions of Single-Output case



Natural extensions of Single-Output case

At the time instant k:

- Guaranteed state set at $k-1$: $\hat{X}_{k-1} = \hat{p}_{k-1} \oplus \hat{H}_{k-1}\mathbf{B}^r$
- Measured output vector: $y_k = \begin{bmatrix} y_{k/1} & \dots & y_{k/n_y} \end{bmatrix}^T$
- Rewrite $W = F\mathbf{B}^{n_x}$, $V = diag(\sigma_1, \dots, \sigma_{n_y})\mathbf{B}^{n_y}$
  - Prediction:

$$\bar{X}_k = A\hat{p}_{k-1} \oplus \begin{bmatrix} A\hat{H}_{k-1} & F \end{bmatrix} \mathbf{B}^{r+n_x} \qquad (14)$$

  - Measurement: n_y strips $\{x \in \mathbb{R}^n : |c_i^T x - y_{k/i}| \leq \sigma_i\}$, $i = 1, \dots, n_y$

Natural extensions of Single-Output case

- Correction:

Intersection with the first strip:

$$\hat{X}_{k/1}(\lambda_1) = \hat{p}_{k/1}(\lambda_1) \oplus \hat{H}_{k/1}(\lambda_1)\mathbf{B}^{r+n_x+1} \tag{15}$$

with $\hat{p}_{k/1}(\lambda_1) = A\hat{p}_{k-1} + \lambda_1(y_{k/1} - c_1^T A\hat{p}_{k-1})$
$\hat{H}_{k/1}(\lambda_1) = \left[(I - \lambda_1 c_1^T)A\hat{H}_{k-1} \quad (I - \lambda_1 c_1^T)F \quad \sigma_1\lambda_1\right]$.

Intersection with the second strip:

$$\hat{X}_{k/2}(\lambda_1, \lambda_2) = \hat{p}_{k/2}(\lambda_1, \lambda_2) \oplus \hat{H}_{k/2}(\lambda_1, \lambda_2)\mathbf{B}^{r+n_x+2} \tag{16}$$

with $\hat{p}_{k/2}(\lambda_1, \lambda_2) = \hat{p}_{k/1}(\lambda_1) + \lambda_2(y_{k/2} - c_2^T \hat{p}_{k/1}(\lambda_1))$ and
$\hat{H}_{k/2}(\lambda_1, \lambda_2) = \left[(I - \lambda_2 c_2^T)\hat{H}_{k/1}(\lambda_1) \quad \sigma_2\lambda_2\right]$.

Correction step

⋮

until the n_y^{th} strip:

$$\hat{X}_{k/n_y}(\lambda_1, ..., \lambda_{n_y}) = \hat{p}_{k/n_y}(\lambda_1, ..., \lambda_{n_y}) \oplus \oplus \hat{H}_{k/n_y}(\lambda_1, ..., \lambda_{n_y}) \mathbf{B}^{r+n_x+n_y} \quad (17)$$

with

$$\hat{p}_{k/n_y}(\lambda_1, ..., \lambda_{n_y}) = \hat{p}_{k_{n_y-1}}(\lambda_1, ..., \lambda_{n_y-1}) + + \lambda_{n_y}(y_{k/n_y} - c_{n_y}^T \hat{p}_{k/n_y-1}(\lambda_1, ..., \lambda_{n_y-1})) \quad (18)$$

and

$$\hat{H}_{k/n_y}(\lambda_1, ..., \lambda_{n_y}) = = \left[(I - \lambda_{n_y} c_{n_y}^T) \hat{H}_{k/n_y-1}(\lambda_1, ..., \lambda_{n_y-1}) \quad \sigma_{n_y} \lambda_{n_y} \right] \quad (19)$$

→Guaranteed state estimation at k: $\hat{X}_k = \hat{p}_k \oplus \hat{H}_k \mathbf{B}^{r+n_x+n_y}$
with $\hat{p}_k = \hat{p}_{k/n_y}$, $\hat{H}_k = \hat{H}_{k/n_y}$.

Illustration of natural extension of Single-Output case

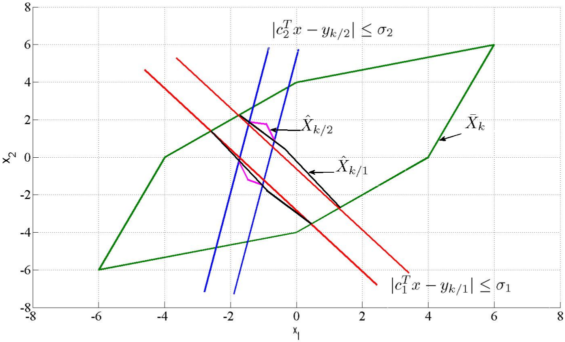
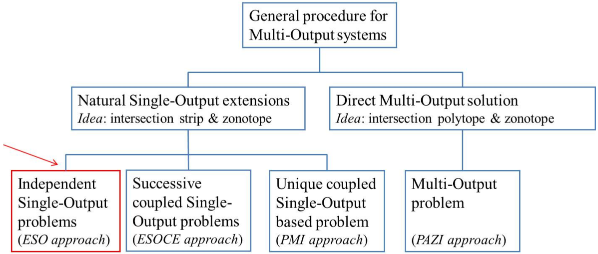


Figure: State estimation of the 2-output system

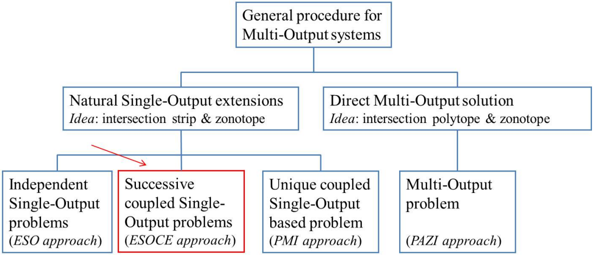
Equivalent Single-Output approach



Approach 1 to compute λ_i: ESO approach

1. For $j = 1, ..., n_y$
   Step j: Using the strip of the measurement $y_{k/j}$ compute λ_j by solving (10);
   End.
2. The guaranteed state estimation is computed by the equation (17) with the known vectors $\lambda_1, \ldots, \lambda_{n_y}$.

Equivalent single-output with coupling effect approach



Approach 2 to compute λ_i: ESOCE approach

1. Step 1: Using the measurement $y_{k/1}$ and (10), compute λ_1;
2. For $j = 2, ..., n_y$

   Step j: Using the measurement $y_{k/j}$ and the previous obtained vectors $\lambda_1, ..., \lambda_{j-1}$, compute λ_j by solving (20).

   End.

$$\max_{\tau,\beta,P,Y_j} \tau$$

subject to BMIs

$$\begin{cases} \tau > 0 \\ \frac{(1-\beta)P}{\sigma_1^2+\ldots+\sigma_j^2+\epsilon} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & \cdots & 0 & B_1 \\ * & F^T F & 0 & \cdots & 0 & B_2 \\ * & * & \sigma_1^2 & \cdots & 0 & B_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \sigma_j^2 & B_{j+2} \\ * & * & * & \cdots & * & P \end{bmatrix} \succeq 0 \end{cases} \quad (20)$$

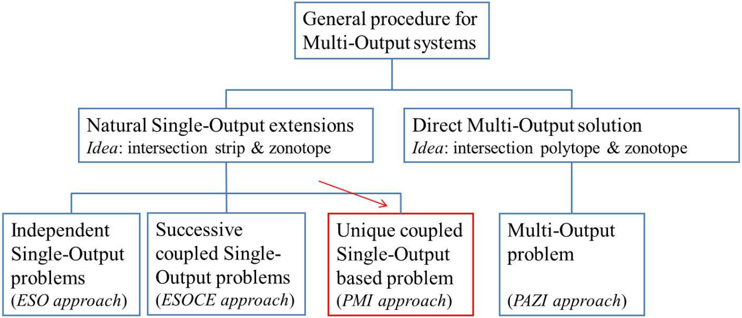
with $\beta \in (0,1)$ and

$$\begin{aligned} B_1 &= ((\prod_{i=1}^{j}(I - \lambda_{j+1-i} c_{j+1-i}^T))A)^T P \\ B_2 &= ((\prod_{i=1}^{j}(I - \lambda_{j+1-i} c_{j+1-i}^T))F)^T P \\ B_3 &= (\prod_{i=1}^{j-1}(I - \lambda_{j+1-i} c_{j+1-i}^T)\sigma_1 \lambda_1)^T P \\ &\vdots \\ B_j &= ((I - \lambda_j c_j^T)(I - \lambda_{j-1} c_{j-1}^T)\sigma_{j-2}\lambda_{j-2})^T P \\ B_{j+1} &= ((I - \lambda_j c_j^T)\sigma_{j-1}\lambda_{j-1})^T P \\ B_{j+2} &= (\sigma_j \lambda_j)^T P \end{aligned} \quad (21)$$

Polynomial Matrix Inequality approach

Approach 3: PMI approach

Computing all $\lambda_1, \ldots, \lambda_{n_y}$ at the same time



$$\max_{\tau,\beta,P,\lambda_1,\ldots,\lambda_{n_y}} \tau$$

subject to PMIs

$$\left\{ \begin{array}{l} \tau > 0 \\ \frac{(1-\beta)P}{\sigma_1^2+\ldots+\sigma_{n_y}^2+\epsilon} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & \cdots & 0 & B_1 \\ * & F^T F & 0 & \cdots & 0 & B_2 \\ * & * & \sigma_1^2 & \cdots & 0 & B_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \sigma_{n_y}^2 & B_{n_y+2} \\ * & * & * & \cdots & * & P \end{bmatrix} \succeq 0 \end{array} \right. \tag{22}$$

with $\beta \in (0,1)$ and

$$\begin{aligned} B_1 &= ((\prod_{i=1}^{n_y}(I - \lambda_{n_y+1-i} c_{n_y+1-i}^T))A)^T P \\ B_2 &= ((\prod_{i=1}^{n_y}(I - \lambda_{n_y+1-i} c_{n_y+1-i}^T))F)^T P \\ B_3 &= (\prod_{i=1}^{n_y-1}(I - \lambda_{n_y+1-i} c_{n_y+1-i}^T)\sigma_1 \lambda_1)^T P \\ &\vdots \\ B_{n_y} &= ((I - \lambda_{n_y} c_{n_y}^T)(I - \lambda_{n_y-1} c_{n_y-1}^T)\sigma_{n_y-2}\lambda_{n_y-2})^T P \\ B_{n_y+1} &= ((I - \lambda_{n_y} c_{n_y}^T)\sigma_{n_y-1}\lambda_{n_y-1})^T P \\ B_{n_y+2} &= (\sigma_{n_y}\lambda_{n_y})^T P \end{aligned} \tag{23}$$

LMI relaxation solution

PMI problem: difficult to solve $\rightarrow$ Sub-optimal solution: use of LMI relaxation [6]

Example: $\min_x(-x_1^2 - x_2^2)$

subject to PMI: $\begin{bmatrix} 1-4x_1x_2 & x_1 \\ x_1 & 4-x_1^2-x_2^2 \end{bmatrix} \succeq 0$

Change of variables:

$y_{10} = x_1,\ y_{01} = x_2,\ y_{20} = x_1^2,\ y_{02} = x_2^2,\ y_{11} = x_1x_2.$

Relaxed optimization problem:

$\min_y(-y_{20} - y_{02})$

subject to LMIs:

$$\begin{cases} \begin{bmatrix} 1 & * & * \\ y_{10} & y_{20} & * \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \succeq 0 \text{ (Moment matrix)} \\ \begin{bmatrix} 1-4y_{11} & * \\ y_{10} & 4-y_{20}-y_{02} \end{bmatrix} \succeq 0 \end{cases} \quad (24)$$

[6] Henrion and Laserre (2006)

LMI relaxation solution

The same relaxation procedure is applied to the PMI problem (22) with the change of variables:

$$\beta = y_{100\ldots0},\ P = \begin{bmatrix} y_{01\ldots0} & \cdots & y_{00\ldots1\ldots0} \\ \ldots & \ldots & \ldots \end{bmatrix},$$

$$\lambda_1^T = \begin{bmatrix} y_{00\ldots1\ldots0} & \ldots \end{bmatrix}, \ldots,\ \lambda_{n_y}^T = \begin{bmatrix} y_{00\ldots1\ldots0} & \cdots & y_{00\ldots01} \end{bmatrix}$$

Solve the optimization problem

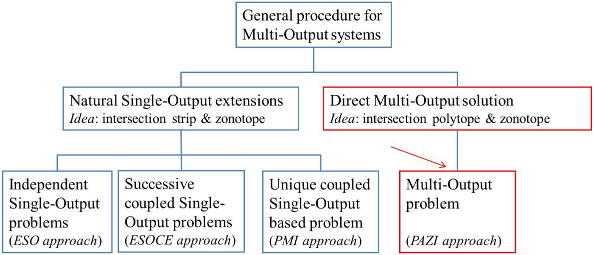
$$\max_{\tau,y} \tau$$

subject to LMIs

$$\begin{cases} M_1 = \begin{bmatrix} 1 & * & * & * \\ y_{10\ldots0} & y_{20\ldots0} & \cdots & * \\ y_{01\ldots0} & y_{11\ldots0} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ y_{00..111} & \cdots & \cdots & y_{00\ldots222} \end{bmatrix} \succeq 0 \\ M_2 \succeq 0 \end{cases} \tag{25}$$

M_2 the equivalent LMI expressions obtained from the PMIs in (22) using the new scalar decision variables.

Direct Multi-Output solution



Previous methods:

- Advantage: direct application of Single-Output case.
- Inconvenient: conservative result due to not computing directly X_{y_k}.

Direct Multi-Output solution:

- Consistent state set created by all the measurements: $X_{y_k} = \{x \in \mathbb{R}^{n_x} : |Cx - y_k| \in V\} \Rightarrow$ a polytope.
- State estimation obtained via the outer approximation of the intersection of zonotope $\bar{X}_k$ and polytope X_{y_k}.

Intersection of a zonotope and a polytope

Proposition: Given:

- zonotope $Z = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$,
- polytope $P = \{x \in \mathbb{R}^n, d \in \mathbb{R}^m : |Cx - d| \leq \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{bmatrix}\}$ $(\sigma_i \in \mathbb{R}^+)$,
- matrix $\Lambda \in \mathbb{R}^{n \times m}$

Define:

- vector $\hat{p}(\Lambda) = p + \Lambda(d - Cp) \in \mathbb{R}^n$,
- matrix $\hat{H}(\Lambda) = \begin{bmatrix}(I - \Lambda C)H & \Lambda\Sigma\end{bmatrix}$, with $\Sigma = diag(\sigma_1, \dots, \sigma_{n_y})$.

Then $Z \cap P \subseteq \hat{Z}(\Lambda) = \hat{p}(\Lambda) \oplus \hat{H}(\Lambda)B^{r+m}$.

Intersection of a zonotope and a polytope

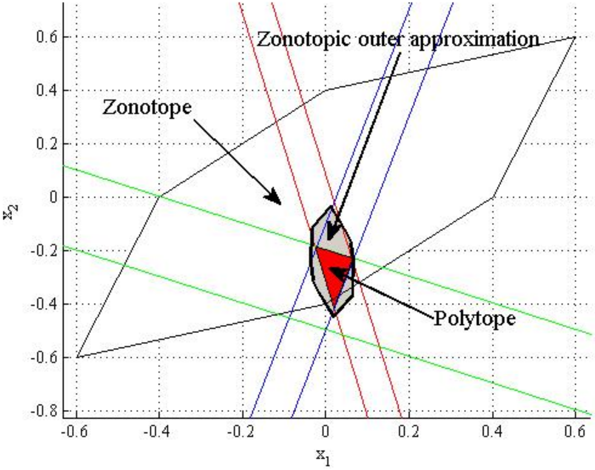


Figure: Zonotopic approximation of the intersection between a zonotope and a polytope

Direct Multi-Output solution

Similar to Single-Output case:
Based on the estimation at $k-1$: $\hat{X}_{k-1} = p \oplus H\mathbf{B}^r$,
the state estimation set at k is:

$$\hat{X}_k(\Lambda) = \hat{p}(\Lambda) \oplus \hat{H}(\Lambda)\mathbf{B}^{r+n_x+n_y} \tag{26}$$

with $\hat{p}(\Lambda) = Ap + \Lambda(y_k - CAp)$
and $\hat{H}(\Lambda) = \left[(I - \Lambda C)\left[AH \quad F\right] \quad \Lambda\Sigma\right]$

- Remark:

- Single-Output case: λ is a vector.
- Multi-Output case: Λ is a matrix.

Direct Multi-Output solution

$\Lambda \in \mathbb{R}^{n_x \times n_y}$ computed to ensure the non-increasing property of the P-radius of the zonotopic guaranteed state estimation:

$$\max_{\tau,\beta,P,Y} \tau$$

subject to BMIs

$$\begin{cases} \tau > 0 \\ \frac{(1-\beta)P}{\sigma_1^2+\ldots+\sigma_{n_y}^2+\epsilon} \succeq \tau I \\ \begin{bmatrix} \beta P & 0 & 0 & A^T P - A^T C^T Y^T \\ * & F^T F & 0 & F^T P - F^T C^T Y^T \\ * & * & \Sigma^T \Sigma & Y^T \Sigma \\ * & * & * & P \end{bmatrix} \succeq 0 \end{cases} \tag{27}$$

with $\beta \in (0,1)$, P and $Y = P\Lambda$ as decision variables.

Example

Consider the following linear discrete-time invariant system:

$$\begin{cases} x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1+0.3\delta \end{bmatrix} x_k + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \omega_k \\ \\ y_k = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} v_k \end{cases} \tag{28}$$

with $\|v_k\|_\infty \leq 1, \|\omega_k\|_\infty \leq 1$, $\|\delta\| \leq 1$.

The initial state belongs to the box $3\mathbf{B}^2$.

Simulation results

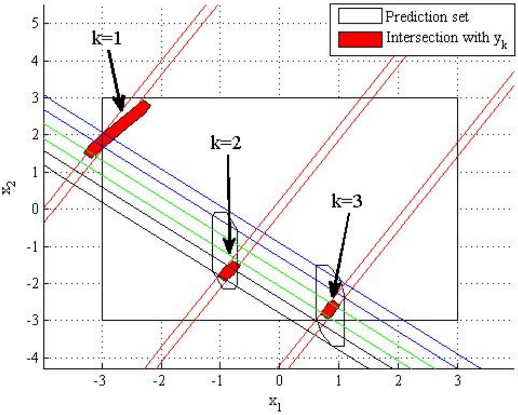
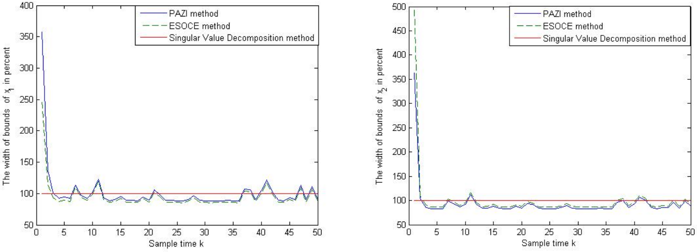


Figure: Evolution of the guaranteed state estimation

-The guaranteed state estimation decreases in time.

Simulation results



(a) x_1 (b) x_2

Figure: Comparison of the state bound's width obtained by ESOCE, PAZI and SVD approaches

Simulation results

Table: Total computation time after 50 samples

Approach	Time(second)
PAZI approach (without off-line LMI optimization included)	0.0468
PAZI approach (with off-line LMI optimization included)	0.2808
ESOCE approach (with off-line optimization included)	1.4664
Singular Value Decomposition approach[7]	1.5444

- Conclusion: PAZI approach offers a low complexity of computation and good precision of estimation.

[7] Combastel (2003)

Outline

Robust tube-based constrained MPC via zonotopic set-membership estimation

Classical output feedback MPC tube-based [8]:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + \omega_k \\ y_k = Cx_k + v_k \end{cases} \tag{29}$$

Luenberger observer:

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k) \\ \hat{y}_k = C\hat{x}_k \end{cases} \tag{30}$$

Difference equation of state estimation error ($\tilde{x}_k = x_k - \hat{x}_k$):
$\tilde{x}_{k+1} = (A - LC)\tilde{x}_k + \omega_k^e$, with $A - LC$ stable and $\omega_k^e = \omega_k - Lv_k$.
Remark: $\tilde{x}_k \in$ invariant set $S_k^e \to \tilde{x}_{k+1} \in$ invariant set $S_{k+1}^e = (A - LC)S_k^e \oplus W^e$ with $W^e = W \oplus (-LV)$

[8] Mayne et al. (2009)

Robust tube-based constrained MPC via zonotopic set-membership estimation

Proposed method: Replace Luenberger observer by the zonotopic guaranteed state estimation.
Consider the center of the zonotopic guaranteed state estimation as the estimated state: $\hat{x}_k = p_k$.

Remark: $\tilde{x}_k = x_k - \hat{x}_k \in$ invariant zonotope S_k^e
$\rightarrow \tilde{x}_{k+1} \in$ invariant $S_{k+1}^e = (I - \Lambda C)AS_k^e \oplus W^e$, with $W^e = (I - \Lambda C)W \oplus (-\Lambda V)$
The size of S_k^e decreases by appropriately choosing Λ.

Robust tube-based constrained MPC via zonotopic set-membership estimation

Nominal system without disturbances:

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \tag{31}$$

To counteract the disturbances, the trajectory is desired to lie close to the nominal trajectory:

$$u_k = \underline{u}_k + Ke_k \tag{32}$$

where $e_k = \hat{x}_k - \underline{x}_k$ satisfies the difference equation:
$e_{k+1} = (A + BK)e_k + w_k^{co}$, where $A + BK$ stable,
$\omega_k^{co} \in W_k^{co} = \Lambda CAS_k^e \oplus \Lambda CW \oplus \Lambda V$

Robust tube-based constrained MPC via zonotopic set-membership estimation

How to define $\underline{u}_k$?
Consider the following cost function at time instant k:

$$V_N(\underline{x}, \underline{u}) = V_f(\underline{x}_N) + \sum_{i=0}^{N-1} l(\underline{x}_i, \underline{u}_i) \tag{33}$$

with $\underline{u} = \{\underline{u}_0, ..., \underline{u}_{N-1}\}$, $l(x, u) = 0.5(x^T Qx + u^T Ru)$, $V_f(x) = 0.5x^T P_f x$,
under the following time varying constraints:

$$\begin{cases} \underline{u}_i \in \underline{U}_{k+i},\ i \in \mathbb{N}_{[0,N-1]} \\ \underline{x}_i \in \underline{X}_{k+i},\ i \in \mathbb{N}_{[0,N-1]} \\ \underline{x}_N \in \underline{X}_f \end{cases} \tag{34}$$

Remark: Sets $\underline{U}_{k+i}$ and $\underline{X}_{k+i}$ are time varying due to the decreasing of the estimation set.

Robust tube-based constrained MPC via zonotopic set-membership estimation

Solution of the optimization problem:

$$(\underline{x}^*(\hat{x},k),\underline{u}^*(\hat{x},k)) = arg\min_{\underline{x},\underline{u}}\{V_N(\underline{x},\underline{u})\} \tag{35}$$

The control law applied to the system:

$$\kappa_N(\hat{x},k) = \hat{u}^*(0,\hat{x},k) + K(\hat{x}_k - \underline{x}^*(\hat{x},k)) \tag{36}$$

with $\hat{u}^*(0,\hat{x},k)$ the first element of the sequence $\underline{u}^*(\hat{x},k)$.

Using the control law (36), $(x,\hat{x})$ is robustly steered to some sets containing the origin, exponentially fast satisfying all constraints[8].

[8] Mayne et al. (2009)

Example

Consider the following linear discrete-time invariant system:

$$x_{k+1} = \begin{bmatrix} 1 & 1.1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + \omega_k$$
$$y_k = \begin{bmatrix} -2 & 1 \end{bmatrix} x_k + v_k$$

- $\|v_k\|_\infty \leq 0.05, \|\omega_k\|_\infty \leq 0.1$.
- $(x, u) \in X \times U$, with:
  $X = \{x \in \mathbb{R}^2 : x_1 \in [-50, 3], x_2 \in [-50, 3]\}$
  $U = \{u \in \mathbb{R} : |u| \leq 9\}$.

Example

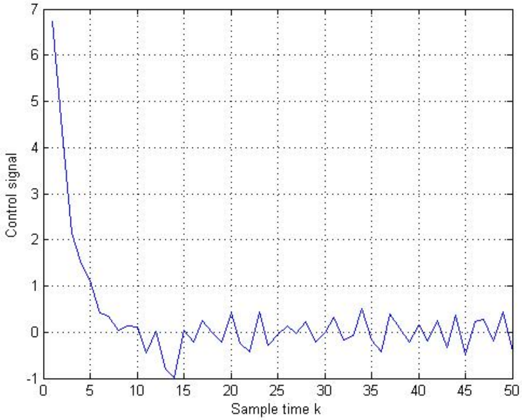


Figure: Control signal

-The control constraint is respected.

Example

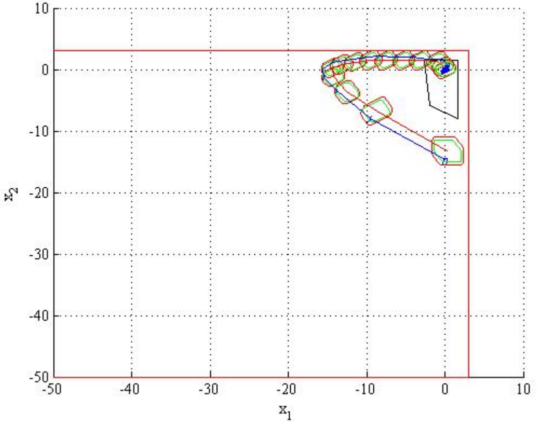


Figure: Tube trajectory of the closed-loop response of the system fulfilling the state constraint

-The state constraints are respected.

Example

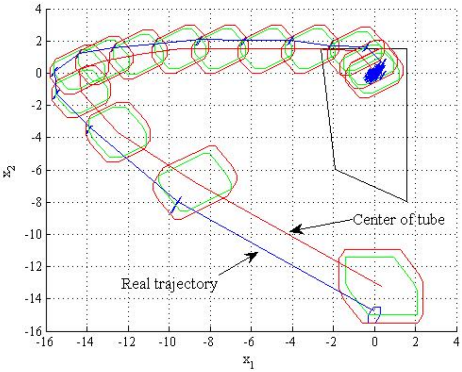


Figure: Zoom of tube trajectory of the closed-loop response of the system

-The size of the tube decreases in time.

-The state converges to an invariant set while respecting the constraints

Example

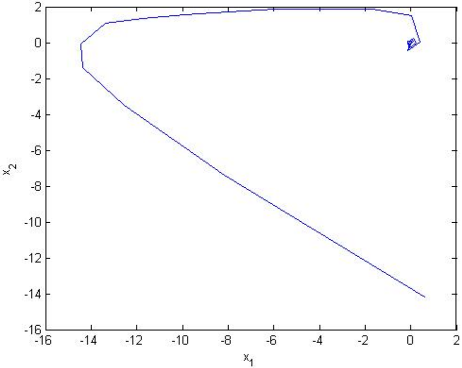


Figure: Closed-loop response of the system

-The system is stable in the sense of Input to State Stability.

Application to a magnetic levitation system

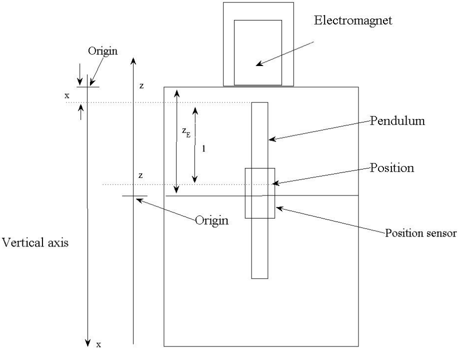


Figure: Magnetic levitation system

Application to a magnetic levitation system

- Consider the continuous-time model[9]:

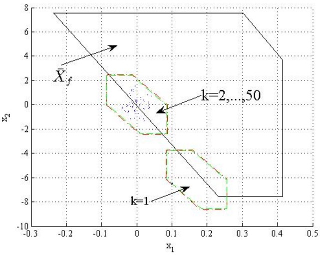
$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{x_0} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{g}{i_0} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases} \tag{37}$$

with $g = 9.81 m/s^2$, $x_0 = 0.019m$, $i_0 = 0.436A$.

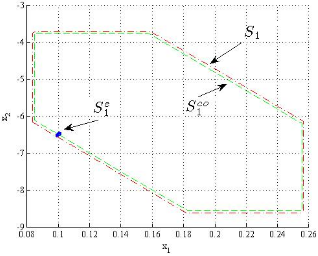
- This model is next discretized with $T_s = 0.1s$.
- Additional model disturbances $\omega_k \in W = \{w \in \mathbb{R} : |w| \leq 1\}$ and measurement noise $v_k \in V = \{v \in \mathbb{R} : |v| \leq 0.05\}$
- Constraints: $X = \{x \in \mathbb{R}^2 : |x_1| \leq 0.5, |x_2| \leq 10\}$, $U = \{u \in \mathbb{R} : |u| \leq 5\}$

[9] Le et al. (2012b)

Application to a magnetic levitation system



(a) Tube trajectory

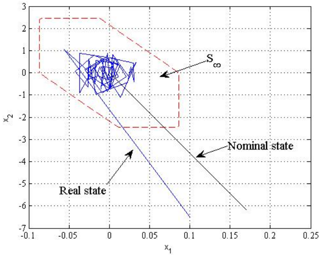


(b) Zoom

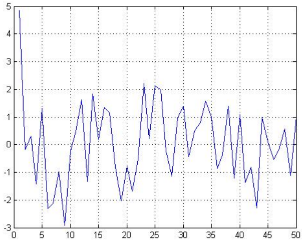
Figure: Tube trajectory of the controlled magnetic levitation system

- In this case, the real system enters to the terminal set after two time instants.

Application to a magnetic levitation system



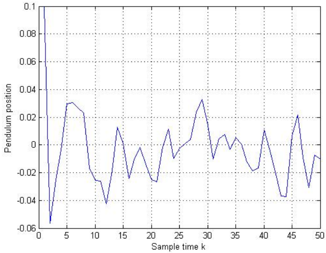
(a) Real state and nominal state of the closed-loop



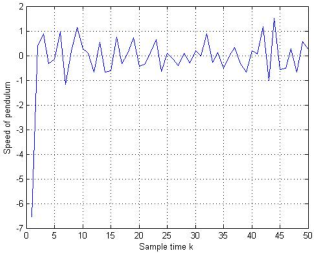
(b) Control signal

- Guarateeing stability of the output feedback system while respecting the constraints.

Application to a magnetic levitation system



(c) Pendulum position



(d) Pendulum speed

Outline

Conclusion

- Estimation
  1. A new method based on the minimization of the P-radius to solve the problem of set-membership estimation using zonotope is presented.
  2. Both Single-Output and Multi-Output systems were considered subject to interval uncertainties, unknown but bounded disturbances and measurement noises.
  3. A good trade-off solution between precision and complexity compared to existing zonotopic set-membership estimation techniques is illustrated.
- Control
  1. The zonotopic set-membership estimation is used in the context of Tube MPC, offering good performance.

Future work

- Output feedback control law based on the P-radius based zonotopic set-membership estimation for systems with interval parametric uncertainties in the presence of disturbances, measurement noises and constraints.
- Extension of the estimation technique to time delay systems.
- Extension to fault diagnosis problem and fault tolerant control.

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