Nonlinear and hybrid reachability analysis in presence of uncertainty.

Nacim RAMDANI,
Univ. Orléans, EA 4229 PRISME à Bourges.

Hybrid Reachability: Motivations
- Analysis of complex dynamical systems
- Reachability-based methods

Nonlinear reachability
- Interval Taylor methods
- Bracketing enclosures
- Software implementation
Hybrid Cyber-Physical Systems

- Interaction discrete + continuous dynamics
- Safety-critical embedded systems
- Networked autonomous systems
Hybrid Cyber-Physical Systems

- Verification
  - Numerical proof
  - Falsification via counter-example
Hybrid Cyber-Physical Systems

- **Modelling → hybrid automaton** (Alur, et al. 1995)
  - Non-linear continuous dynamics
  - Bounded uncertainty

\[
H = (Q, D, \mathcal{P}, \Sigma, \mathcal{A}, \text{Inv}, \mathcal{F}),
\]

**Continuous dynamics**

- \( \dot{x}(t) = f_q(x, p, t) \)
- \( \text{Inv}(q) : \nu_q(x(t), p, t) < 0 \)

**Discrete dynamics**

- \( \mathcal{A} \ni e : (q \rightarrow q') = (q, \text{guard}, \sigma, \rho, q') \)
- \( \gamma_e(x(t), p, t) = 0 \)

\[
t_0 \leq t \leq t_N, \quad x(t_0) \in X_0 \subseteq \mathbb{R}^n, \quad p \in \mathcal{P}
\]
Example: bouncing ball
Example: bouncing ball
Hybrid Cyber-Physical Systems

**Verification**
- **Modelling:**
- **Property specification:**

**Verification algorithm:**
- Hybrid / Continuous reachability
Continuous reachability

\[ R([t_0, t]; X_0) = \left\{ x(\tau), \ t_0 \leq \tau \leq t \mid \dot{x}(\tau) = f(x, p, \tau) \land x(t_0) \in X_0 \land p \in P \right\} \]

- Set integration
  - Interval Taylor methods
  - Bracketing enclosures
Continuous reachability

\[ \mathcal{R}([t_0, t]; X_0) = \left\{ x(\tau), \quad t_0 \leq \tau \leq t \mid \dot{x}(\tau) = f(x, p, \tau) \land x(t_0) \in X_0 \land p \in P \right\} \]

- Set integration
  - Interval Taylor methods
  - Bracketing enclosures
Hybrid Cyber-Physical Systems

- Hybrid reachability
  - Continuous reachability
  - Guard conditions, jumps & resets
Guaranteed event detection & localization

- An interval constraint propagation approach
  - (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2011)

Time grid $t_0 < t_1 < t_2 < \cdots < t_N$

Compute $[t^*, \bar{t}^*] \times [x_j^*]$
Guaranteed event detection & localization

- An interval constraint propagation approach
  - (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2011)

\[ \text{Time grid} \rightarrow \quad t_0 < t_1 < t_2 < \cdots < t_N \]

- \[ [x](t) = \text{Interval Taylor Series (ITS)}(t, [x_j], [\dot{x}_j]) \]
- \[ \gamma([x](t)) = 0 \]

\[ \Rightarrow \gamma \circ \text{ITS}(t, x_j, [\ddot{x}_j]) \rightarrow \psi(t, x_j) \]

Solve CSP \( ([t_j, t_{j+1}] \times [x_j], \psi(\ldots) \ni 0) \)
Guaranteed event detection & localization

- An interval constraint propagation approach
  - (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2011)
Hybrid Reachability Analysis

- Detecting and localizing events
  - Improved and enhanced version
    - (Maïga, et al., IEEE CDC 2013, ECC 2014)
Hybrid Reachability Analysis

- Detecting and localizing events
  - Improved and enhanced version
    - (Maïga, et al., IEEE CDC 2013, ECC 2014)
Hybrid Reachability Analysis

- Detecting and localizing events
- Improved and enhanced version
  - (Maïga, et al., IEEE CDC 2013, ECC 2014)

\[ \sigma = [0, 0.01] \text{ and } h=0.5 \]

CPU times=87s with HC4 contractor
CPU times> 1h without HC4 contractor
Verification of Hybrid Systems

- Verification:
  - Reachability of a forbidden area
Aircraft traffic control [Tomlin, et al.]

Collision possible!

Verification of Hybrid Systems

Disturbance

Reachable sets

Time
Bounded Model Checking

- Can the system reach an \textit{unsafe} state within \( k \) (discrete or continuous) transition steps?
- Check \textit{satisfiability} of a \textit{SAT Mod ODE} formula

\[ \Phi_k := init[0] \land trans[0, 1] \land \cdots \land trans[k - 1, k] \land target[k] \]
Bounded Model Checking

- Can the system reach an **unsafe** state within \( k \) (discrete or continuous) transition steps?
- Check **satisfiability** of a SAT Mod ODE formula

\[
\Phi_k := \text{init}[0] \land \text{trans}[0, 1] \land \cdots \land \text{trans}[k-1, k] \land \text{target}[k]
\]

**init** =

\[
\begin{align*}
-10 \leq \vartheta_o & \leq 20 \land c = 0 \\
\land & \left( 19 \leq \vartheta_i \leq 25 \land \neg on \right) \\
\lor & \left( 15 \leq \vartheta_i \leq 21 \land on \right)
\end{align*}
\]

**trans** =

\[
\begin{align*}
\neg on \land on' \land \vartheta_i & \leq 19 \land c \leq 0.04 \\
\land \vartheta_o' & = \vartheta_i \land \vartheta_o = \vartheta_o \land c' = c \\
\lor ( on \land \neg on' \land \vartheta_i & \geq 21 \\
\land \vartheta_o' & = \vartheta_i \land \vartheta_o = \vartheta_o \land c' = c ) \\
\lor ( \neg on \land \neg on' \\
\land \vartheta_o' & = -0.1(\vartheta_i - \vartheta_o) \\
\land \vartheta_o' & = -0.05c \\
\land ( \vartheta_i \geq 19 \land c' \geq 0.04 ) \land \vartheta_o' = \vartheta_o ) \\
\lor ( on \land on' \\
\land \vartheta_o' & = 0.2 \cdot 35 - 0.3 \vartheta_i + 0.1 \vartheta_o \\
\land \vartheta_o' & = 0.01 - 0.05c \\
\land \vartheta_i \leq 21 \land \vartheta_o' = \vartheta_o )
\end{align*}
\]

**target** =

\[
(c > 0.1)
\]
SAT mod ODE

- **Model**: init → definition of variables.
  trans[k,k+1] → transition dynamics.

- **Property**: prop

**SAT solvers check the following formulas:**

- init ∧ ¬prop
- init ∧ trans[0,1] ∧ ¬prop
- init ∧ trans[0,1] ∧ trans[1,2] ∧ ¬prop
- init ∧ trans[0,1] ∧ trans[1,2] ∧ trans[2,3] ∧ ¬prop ...

- If one formula is satisfiable → Property is violated!
Example: 2-tanks system

For $x_2 > k_3$:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
\frac{k_1 - k_2 \sqrt{x_1 - x_2 + k_3}}{k_2 \sqrt{x_1 - x_2 + k_3 - k_4 \sqrt{x_2}}}
\end{bmatrix}
$$

For $x_2 \leq k_3$:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
\frac{k_1 - k_2 \sqrt{x_1}}{k_2 \sqrt{x_1 - k_4 \sqrt{x_2}}}
\end{bmatrix}
$$

$k_1 = 0.75$, $k_2 = 1$, $k_3 = 0.5$, $k_4 = 1$
Example : 2-tanks system
E non reachable from D. [Eggers, Ramdani, Nedialkov, Fränzle, 2011]
iSAT-ODE: Proof in 260s CPU 2.4 GHz AMD Opteron
Proof: a trajectory starting in $A$, stays in $A$ during $[\tau, 2\tau]$

SAT mod ODE formula
Target:
Non reached at $2\tau$
or left $A$ during $[\tau, 2\tau]$

If UNSAT, recurrence, time-invariance, infinite time property.

Reference:
[Podelski et Wagner, 2007] [Eggers, Ramdani, Nedialkov, Fränzle, 2011]
Region stability

[Eggers, Ramdani, Nedialkov, Fränzle, 2011]
iSAT-ODE: proof in 150s CPU 2.4 GHz AMD Opteron
Hybrid systems synthesis

Hybrid reachability analysis

- Verification
- Synthesis
- Set-theoretic estimation
Outline

Hybrid Reachability: Motivations
- Analysis of complex dynamical systems
- Reachability-based methods

Nonlinear reachability
- Interval Taylor methods
- Bracketing enclosures
- Software implementation
Nonlinear Set Integration

Non-guaranteed integration method ...

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]
Guaranteed set integration with Taylor methods

- (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]
\]
Nonlinear Set Integration

- Guaranteed set integration with Taylor methods
  - (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \ p \in [p]
\]

- Proof of existence
- Yield a priori solution \([\tilde{x}_j]: \forall \tau \in [t_j, t_{j+1}] \quad x(\tau) \in [\tilde{x}_j]\)
Guaranteed set integration with Taylor methods

\( (\text{Moore,66}) \ (\text{Lohner,88}) \ (\text{Rihm,94}) \ (\text{Berz,98}) \ (\text{Nedialkov,99}) \)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]
\]

\[
[x_j] + [0, h]f([\tilde{x}_j]) \subseteq [\tilde{x}_j]
\]
Nonlinear Set Integration

- Guaranteed set integration with Taylor methods
  - (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]
\]

*a priori* enclosure (*entrée* : \([x_j], h, \alpha* ; *sortie* : \([\tilde{x}_j])\)

1. Initialisation : \([\tilde{x}_j] := [x_j] + [0, h] f ([x_j]) ;
2. tant que \(( [x_j] + [0, h] f ([\tilde{x}_j]) \not\subset [\tilde{x}_j])\)

\[
[\tilde{x}_j] := [\tilde{x}_j] + [-\alpha, \alpha] |[\tilde{x}_j]|
\]

\[
h := h/2
\]

\text{fin}

---

Guaranteed set integration with Taylor methods

- (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]
\]

Time grid \( t_0 < t_1 < t_2 < \cdots < t_N \)

Compute tight enclosure \([x_{j+1}] \supset x(t_{j+1})\)

\[
[x_{j+1}] = [x_j] + \sum_{i=1}^{k-1} (t_{j+1} - t_j)^i f^{[i]}([x_j], [p]) + (t_{j+1} - t_j)^k f^{[k]}([\tilde{x}_j], [p])
\]
Nonlinear Set Integration
Guaranteed set integration with Taylor methods

(Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[
\dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p]
\]

Time grid $\rightarrow$ $t_0 < t_1 < t_2 < \cdots < t_N$

Analytical solution for $[x](t)$, $t \in [t_j, t_{j+1}]$

\[
[x](t) = [x_j] + \sum_{i=1}^{k-1} (t - t_j)^i f[i]([x_j], [p]) + (t - t_j)^k f[k]([\tilde{x}_j], [p])
\]
Nonlinear Set Integration

Guaranteed set integration with Taylor methods

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]
Nonlinear Set Integration

Guaranteed set integration with Taylor methods

(Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \; x(t_0) \in [x_0], \; p \in [p] \]

\[
\begin{align*}
    f[1] &= x^{(1)} = f \\
    f[2] &= \frac{1}{2}x^{(2)} = \frac{1}{2}\frac{df}{dx}f \\
    f[i] &= \frac{1}{i!}x^{(i)} = \frac{1}{i}\frac{df[i-1]}{dx}f, \quad i \geq 2
\end{align*}
\]
Nonlinear Set Integration

- Wrapping effect
  - (Moore, 66)
Guaranteed set integration with Taylor methods


\[ \dot{x}(t) = f(x, p, t), \quad t_0 \leq t \leq t_N, \quad x(t_0) \in [x_0], \quad p \in [p] \]

Mean-value approach

\[ [x](t) \in \{v(t) + A(t)r(t) \mid v(t) \in [v](t), \ r(t) \in [r](t)\}. \]
Guaranteed set integration with Taylor methods

• (Moore, 66) (Lohner, 88) (Rihm, 94) (Berz, 98) (Nedialkov, 99)

Complexity

• Work per step is of polynomial complexity
  • Computing Taylor coefficients $\rightarrow o(k^2)$
  • Linear algebra $\rightarrow o(n^3)$

In practice: Obtaining Taylor coefficients ...

• FADBAD++ ([www.fadbad.com](http://www.fadbad.com))
  Flexible Automatic differentiation using templates and operator overloading in C++
An Interval Solver for Initial Value Problems in Ordinary Differential Equations

Ned Nedialkov
nedialk@mcmaster.ca

VNODE-LP is a C++ package for computing bounds on solutions in IVPs for ODEs. In contrast to traditional ODE solvers, which compute approximate solutions, this solver tries to prove that a unique solution to a problem exists and then computes bounds that contain this solution. Such bounds can be used to help prove a theoretical result, check if a solution satisfies a condition in a safety-critical calculation, or simply to verify the results produced by a traditional ODE solver.

This package is a successor of the VNODE package of N. Nedialkov. A distinctive feature of the present solver is that it is developed entirely using Literate Programming. As a result, the correctness of VNODE-LP's implementation can be examined easier than the correctness of VNODE: the theory, documentation, and source code are produced from the same CWEB files.
Comparison theorems for differential inequalities

- Müller’s existence theorem (1936)

\[
\begin{align*}
\forall t \in [t_0, t_N], \forall x \in \mathbb{D}, \forall p \in \mathbb{P} \\
\forall i \min_{x_i = \omega_i} f_i(x, p, t) &\geq D^\pm \omega_i(t) \\
\forall i \max_{x_i = \Omega_i} f_i(x, p, t) &\leq D^\pm \Omega_i(t) \\
\omega(t_0) &\leq x(t_0) \leq \Omega(t_0)
\end{align*}
\Rightarrow
\begin{align*}
\forall t \in [t_0, t_N], \\
\omega(t) &\leq x(t) \leq \Omega(t)
\end{align*}
\]

- Bracketing systems
**Comparison theorems for differential inequalities**

- **Bracketing systems**

![Graph showing state over time with maximal and minimal solutions marked](image)
Müller’s theorem
Müller’s theorem
Müller’s theorem

\[
\max_{x_1} f_2(x_1, \Omega_2)
\]
Müller’s theorem
Müller’s theorem
Bracketing systems

Dynamics of ...

\[
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \overline{x}_{1,0}] \subset \mathbb{R}, \quad p \in [\underline{p}, \overline{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \overline{x}_{2,0}] \subset \mathbb{R},
\end{aligned}
\]

If \( \forall t \geq t_0, \forall x(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2, \forall p \in [\underline{p}, \overline{p}] \),

\[
\frac{\partial f_1}{\partial x_2} > 0 \land \frac{\partial f_1}{\partial p} > 0
\]

then \( f_1(\omega_1, \omega_2, p) \leq f_1(\omega_1, x_2, p, t) \) and \( f_1(\Omega_1, x_2, p, t) \leq f_1(\Omega_1, \Omega_2, \overline{p}) \)

\( \dot{\omega}_1(t) \equiv f_1(\omega_1, \omega_2, p) \) and \( f_1(\Omega_1, \Omega_2, \overline{p}) \equiv \dot{\Omega}_1(t) \)
Bracketing systems

Dynamics of ...

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, p, t), \quad x_1(t_0) \in [x_{1,0}, \bar{x}_{1,0}] \subset \mathbb{R}, \quad p \in [\underline{p}, \bar{p}] \quad t \geq t_0 \\
\dot{x}_2 &= f_2(x_1, x_2, p, t), \quad x_2(t_0) \in [x_{2,0}, \bar{x}_{2,0}] \subset \mathbb{R},
\end{align*}
\]

If \( \forall t \geq t_0, \forall x(t) \in [\omega(t), \Omega(t)] \subset \mathbb{R}^2, \forall p \in [\underline{p}, \bar{p}] \),

\[
\frac{\partial f_1}{\partial x_2} > 0 \land \frac{\partial f_1}{\partial p} > 0
\]

then

\[
\begin{align*}
\dot{\omega}_1(t) &\equiv f_1(\omega_1, \omega_2, p) \\
\dot{\Omega}_1(t) &\equiv \bar{f}_1(\Omega_1, \Omega_2, \bar{p})
\end{align*}
\]

and

\[
\begin{align*}
f_1(\omega_1, x_2, p, t) &\leq f_1(\omega_1, x_2, p, t) \\
f_1(\Omega_1, x_2, p, t) &\leq f_1(\Omega_1, x_2, p, t)
\end{align*}
\]

and

\[
\begin{align*}
f_1(\Omega_1, \Omega_2, \bar{p}) &\equiv \dot{\Omega}_1(t)
\end{align*}
\]
Comparison theorems for differential inequalities

- Müller’s existence theorem (1936)

\[
\begin{align*}
\forall t \in [t_0, t_N], \forall x \in D, \forall p \in P \quad & \\
\forall i \min_{x_i=\omega_i} f_i(x, p, t) \geq D^+ \omega_i(t) \quad & \\
\forall i \max_{x_i=\Omega_i} f_i(x, p, t) \leq D^- \Omega_i(t) \quad & \\
\omega(t_0) \leq x(t_0) \leq \Omega(t_0) \quad & \Rightarrow \\
\forall t \in [t_0, t_N], \quad & \\
\omega(t) \leq x(t) \leq \Omega(t) \\
solution exists
\end{align*}
\]

- Bracketing systems: coupled EDOs

\[
\begin{align*}
\dot{\omega}(t) &= f(\omega, \Omega, p, \overline{p}, t), \quad \omega(t_0) = x_0 \\
\dot{\Omega}(t) &= f(\omega, \Omega, p, \overline{p}, t), \quad \Omega(t_0) = \overline{x}_0
\end{align*}
\]
Bracketing systems

- Example: Mitogen-Activated Protein Kinase (Sontag, 2005)
Bracketing systems

Example: Mitogen-Activated Protein Kinase (Sontag, 2005)

\[
\begin{align*}
\dot{x}_1 &= -\frac{v_2 x_1}{k_2 + x_1} + v_0 u + v_1 \\
\dot{x}_2 &= \frac{v_6 (y_{tot} - x_2 - x_3)}{k_6 + (y_{tot} - x_2 - x_3)} - \frac{v_3 x_1 x_2}{k_3 + x_2} \\
\dot{x}_3 &= \frac{v_4 x_1 (y_{tot} - x_2 - x_3)}{k_4 + (y_{tot} - x_2 - x_3)} - \frac{v_5 x_3}{k_5 + x_3} \\
\dot{x}_4 &= \frac{v_10 (z_{tot} - x_4 - x_5)}{k_{10} + (z_{tot} - x_4 - x_5)} - \frac{v_7 x_3 x_4}{k_7 + x_4} \\
\dot{x}_5 &= \frac{v_8 x_3 (z_{tot} - x_4 - x_5)}{k_8 + (z_{tot} - x_4 - x_5)} - \frac{v_9 x_5}{k_9 + x_5} \\
u &= g x_5
\end{align*}
\]
**Nonlinear Set Integration**

**Bracketing systems**

- Example: Mitogen-Activated Protein Kinase (Sontag, 2005)

\[
\begin{align*}
\dot{x}_1 &= -\frac{v_2 x_1}{k_2 + x_1} + v_0 u + v_1 \\
\dot{x}_2 &= \frac{v_6 (y_{tot} - x_2 - x_3)}{k_6 + (y_{tot} - x_2 - x_3)} - \frac{v_3 x_1 x_2}{k_3 + x_2} \\
\dot{x}_3 &= \frac{v_4 x_1 (y_{tot} - x_2 - x_3)}{k_4 + (y_{tot} - x_2 - x_3)} - \frac{v_5 x_3}{k_5 + x_3} \\
\dot{x}_4 &= \frac{v_{10} (z_{tot} - x_4 - x_5)}{k_{10} + (z_{tot} - x_4 - x_5)} - \frac{v_7 x_3 x_4}{k_7 + x_4} \\
\dot{x}_5 &= \frac{v_8 x_3 (z_{tot} - x_4 - x_5)}{k_8 + (z_{tot} - x_4 - x_5)} - \frac{v_9 x_5}{k_9 + x_5} \\
\dot{x}_1 &= -\frac{v_2 x_1}{k_2 + x_1} + v_0 u + v_1 \\
\dot{x}_2 &= \frac{v_6 (y_{tot} - x_2 - x_3)}{k_6 + (y_{tot} - x_2 - x_3)} - \frac{v_3 x_1 x_2}{k_3 + x_2} \\
\dot{x}_3 &= \frac{v_4 x_1 (y_{tot} - x_2 - x_3)}{k_4 + (y_{tot} - x_2 - x_3)} - \frac{v_5 x_3}{k_5 + x_3} \\
\dot{x}_4 &= \frac{v_{10} (z_{tot} - x_4 - x_5)}{k_{10} + (z_{tot} - x_4 - x_5)} - \frac{v_7 x_3 x_4}{k_7 + x_4} \\
\dot{x}_5 &= \frac{v_8 x_3 (z_{tot} - x_4 - x_5)}{k_8 + (z_{tot} - x_4 - x_5)} - \frac{v_9 x_5}{k_9 + x_5} \\
u &= \frac{g x_5}{g x_5}
\end{align*}
\]
Nonlinear hybridization


\[ \dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \quad p_i \in [p_i, \bar{p}_i] \]

\[ g_i(.) = \frac{\partial f}{\partial p_i}(.) \]
Nonlinear hybridization


\[ \dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \quad p_i \in [p_i, \bar{p}_i] \]

\[ g_i(\cdot) = \frac{\partial f}{\partial p_i}(\cdot) \]
Nonlinear hybridization


\[ \dot{x} = f(x, p_1, p_2, t) \quad x(t_0) \in [x_0, \bar{x}_0] \subset \mathbb{R}, \quad p_i \in [p_i, \bar{p}_i] \]
Nonlinear hybridization

Monotone order-preserving systems

- Preserve ordering on initial conditions.

\[ x(t_0) \prec y(t_0) \Rightarrow \forall t \geq t_0 \quad x(t) \prec y(t) \quad \prec \in \{<, \leq, \geq, >\} \]
Monotone order-preserving systems

- Graphical test: monotone wrt orthant cones (Kunze & Siegel, 1999)

\[
\text{if } \exists D = \text{diag}[-1^{\varepsilon_1}, \ldots, (-1)^{\varepsilon_n}], \varepsilon_i \in \{0, 1\}
\]

s.t. \( x(t, x_0, t_0) \) and \( y(t, y_0, t_0) \) satisfy

\[
Dy_0 \geq Dx_0 \Rightarrow Dy(t, y_0, t_0) \geq Dx(t, x_0, t_0) \forall t \geq t_0.
\]
Nonlinear Set Integration

Monotone order-preserving systems

- Graphical test: monotone wrt orthant cones (Kunze & Siegel, 1999)

\[
\begin{align*}
\text{if } & \exists D = \text{diag}[-1^{\varepsilon_1}, \ldots, -1^{\varepsilon_n}], \varepsilon_i \in \{0, 1\} \\
\text{s.t } & x(t, x_0, t_0) \text{ and } y(t, y_0, t_0) \text{ satisfy} \\
Dy_0 \geq Dx_0 \Rightarrow Dy(t, y_0, t_0) \geq Dx(t, x_0, t_0) \forall t \geq t_0.
\end{align*}
\]

\[
\begin{aligned}
\dot{x}_1 &= -(v_2x_1)/(k_2 + x_1) + v_0x_5 + v_1 \\
\dot{x}_2 &= (v_6(y_{\text{tot}} - x_2 - x_3))/(k_6 + (y_{\text{tot}} - x_2 - x_3)) - (v_3x_1x_2)/(k_3 + x_2) \\
\dot{x}_3 &= (v_4x_1(y_{\text{tot}} - x_2 - x_3))/(k_4 + (y_{\text{tot}} - x_2 - x_3)) - (v_5x_3)/(k_5 + x_3) \\
\dot{x}_4 &= (v_{10}(z_{\text{tot}} - x_4 - x_5))/(k_{10} + (z_{\text{tot}} - x_4 - x_5)) - (v_7x_3x_4)/(k_7 + x_4) \\
\dot{x}_5 &= (v_8x_3(z_{\text{tot}} - x_4 - x_5))/(k_8 + (z_{\text{tot}} - x_4 - x_5)) - (v_9x_5)/(k_9 + x_5)
\end{aligned}
\]
Nonlinear Set Integration

Nonlinear hybridization & Monotone systems

(Ramdani, et al., Nonlinear Analysis Hybrid Systems 2010)
Nonlinear hybridization & Monotone systems

- (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2010)
- Non-coupled bracketing systems
Nonlinear hybridization & Monotone systems

- (Ramdani, et al., Nonlinear Analysis Hybrid Systems 2010)
- Non-coupled bracketing systems
Nonlinear Set Integration

- Interval Taylor methods vs Bracketing systems

\[ \dot{x} = -p_4 x - \frac{p_1 x}{1 + p_2 y} + p_3 y + 0.1 \]
\[ \dot{y} = p_4 x - p_3 y \]

\[ x(0) \in [1, 1.2], \ y(0) \in [0.8, 1] \]
\[ p_1 \in [0.8, 1], \ p_2 \in [1.0, 1.2], \ p_3 \in [0.3, 0.5], \]
\[ p_4 \in [0.20, 0.25] \]
Nonlinear Set Integration

**Interval Taylor methods vs Bracketing systems**

\[
\dot{x} = y, \quad \dot{y} = -x,
\]

\[x(0), y(0) \in [1, 2]\]
Nonlinear Set Integration

IOLAVABE: iSAT-ODE Layer Around VNODE-LP and Bracketing Enclosures

About

The IOLAVABE library encapsulates the part of the iSAT-ODE tool that handles the generation of ODE enclosures using VNODE-LP and bracketing systems.

IOLAVABE is made available here solely for scientific research.

Detailed licensing information can be found in the LICENSE file inside the source code archive. IOLAVABE depends on and the archive file contains modified versions of VNODE-LP (itself including a copy of FADBAD++) and of filib++. The unmodified versions can be found in the bundled archive as well. Please note the licensing information shipped with these and all indirectly or directly used libraries as well (you will find pointers to the respective terms of use in the INSTALL or LICENSE file or in your system’s package management system).

Installation instructions are to be found in the INSTALL file, and a list of changes with respect to earlier releases can be found in the changelog file.

Contact the author: Andreas Eggers

https://seshome.informatik.uni-oldenburg.de/eggers/iolavabe.php
generates **on-the-fly** hybrid bracketing systems, i.e. tries to re-start bracketing system when monotonicity changes.

uses subordinate local optimization to compute **signs of partial derivatives** on subranges to improve bracketing.
**Typical results:** Taylor methods vs Bracketing systems
- harmonizes bracketing and direct enclosures, i.e. synchronizes time step,

- often intersects enclosures and reinitializes methods
stores Taylor coefficients to recompute «refined» enclosures at intermediate steps.

\[ \dot{x} = -y, \quad \dot{y} = 0.6 \cdot x, \quad x_0 \in [1, 2], \quad y_0 \in [4, 6], \quad t_1 = 1.6 \]
detects independent group of ODEs

detects when flow invariants are being left
can contract pre- & post-box using forward and backward deductions
algorithm's parameters are exposed to the outside

parsers for ODEs and flow invariants offer string interface
Deux méthodes complémentaires

Construction des systèmes englobants à la volée

Eggers, 2011 (Univ. Oldenburg)

Utilisation en parallèle et intersection des encadrements

[Eggers, Ramdani, Nedialkov, Fränzle, 2012]
IOLAVABE:
the iSAT-ODE layer around VNODE-LP and bracketing enclosures
gives a high-level interface for generating enclosures of ODE constraints

Source code available for not-for-profit civilian scientific research: try it!
Hybrid Reachability: Motivations
- Analysis of complex dynamical systems
- Reachability-based methods

Nonlinear reachability
- Interval Taylor methods
- Bracketing enclosures
- Software implementation
All papers on
http://lune.bourges.univ-orleans.fr/ramdani

Thank you!

Questions?