Robust Nonlinear Model Predictive Control for Regulation of Microalgae Culture in a Continuous Photobioreactor

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Context

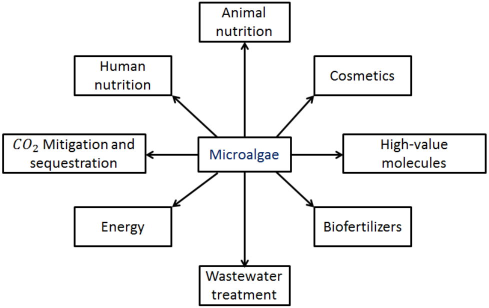
biological & chemical characteristics ⟶ growing demand of organic products

Applications

- food;
- pharmacology, chemistry;
- sustainable environment.

⇓

Industrial success



Industrial regulation

- partial pressure of dissolved oxygen and dissolved carbon dioxide;
- pH and temperature.

Nonlinear control strategies

- optimization-based approaches [Abdollahi and Dubljevic, 2012];
- adaptive approaches [Mailleret et al., 2004];
- sliding mode control [Selisteanu et al., 2007];
- Generic Model Control (nonlinear PI) [Jenzsch et al., 2006];
- exact linearization approach [Ifrim et al., 2013, Tebbani et al., 2015];
- backstepping approach [Dochain and Perrier, 2004].

Choice (advanced & optimal)

Nonlinear Model Predictive Control [Camacho and Bordons, 2004, Kerrigan and Maciejowski, 2004]

Assumptions

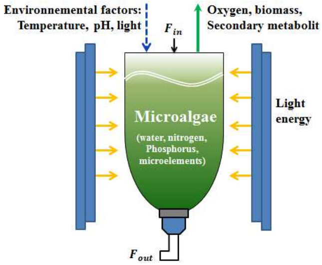
- Photobioreactor in continuous mode ($F_{in} = F_{out}$);
- without any additional biomass in the feed;
- neglecting the effect of gas exchanges.

Mass balances [Masci et al., 2010]

$$\left\{ \begin{array}{l} \dot{X}(t) = \mu(Q,I)\,X(t) - D(t)X(t) \\ \dot{Q}(t) = \rho(S) - \mu(Q,I)\,Q(t) \\ \dot{S}(t) = (S_{in} - S(t))D(t) - \rho(S)X(t) \end{array} \right. \quad (1)$$

$$\text{with :} \left\{ \begin{array}{l} \rho(S) = \rho_m \frac{S}{S+K_s} \quad \textbf{(Monod)} \\ \mu(Q,I) = \bar{\mu}(1 - \frac{K_Q}{Q})\mu_I \quad \textbf{(Droop)} \\ \mu_I = \frac{I}{I+K_{sI}+\frac{I^2}{K_{iI}}} \quad \textbf{(Haldane)} \end{array} \right. \quad (2)$$

Model parameters [Goffaux and Vande Wouwer, 2008], [Munoz-Tamayo et al., 2014].



- X: biomass concentration;
- Q: internal quota of azote;
- S: substrate concentration;
- D: dilution rate;
- I: light intensity;
- S_{in}: inlet substrate concentration;

Dynamical model

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), \theta) \\ y(t) = X(t) \\ x(t_0 = 0) = x_0 \end{cases} \quad (3)$$

with:

$$\begin{cases} x = [X\ Q\ S]^\top,\ u = D, \\ f = \begin{bmatrix} \mu(Q,I)X - DX \\ \rho(S) - \mu(Q,I)Q \\ (S_{in} - S)D - \rho(S)X \end{bmatrix} \\ \theta = \begin{bmatrix} \rho_m & K_s & \bar{\mu} & K_Q & K_{sI} & K_{iI} \end{bmatrix}^\top \end{cases} \quad (4)$$

Constraints [Bernard and Gouzé, 1995]

$$\begin{cases} X \geq 0, \\ K_Q \leq Q \leq \frac{\rho_m}{\bar{\mu}\mu_I} + K_Q, \\ 0 \leq S \leq S_{in}, \\ D \geq 0 \end{cases} \quad (5)$$

Steady state characterization

given $X \implies$ characterize the corresponding Q, S & D

- At the equilibrium: $f(x(t), u(t), \theta) = 0$

$$\begin{cases} D = \pi_1(Q, I) \\ S = \pi_2(Q, I) \\ X = \pi_3(Q, S, I) \end{cases} \quad (6)$$

- After developments (taking Q as an unknown variable):

$$Q_{1,2} = \frac{\bar{\mu}\mu_I(S_{in} + K_s) + (\rho_m + \bar{\mu}\mu_I K_Q)X \pm \sqrt{\Delta}}{2\bar{\mu}\mu_I X} \quad (7)$$

with

$$\Delta = \Big((\rho_m + \bar{\mu}\mu_I K_Q)X - \bar{\mu}\mu_I(S_{in} + K_s)\Big)^2 + 4\bar{\mu}\mu_I K_s \rho_m X > 0 \quad (8)$$

- $Q_1 > \frac{\rho_m}{\mu_I \bar{\mu}} + K_Q \implies Q_1 \notin$ admissible solution.

Control problem

Find a stabilizing control strategy that

- minimizes the objective functional ($X \longrightarrow X^r$ & $D \longrightarrow D^r$);
- satisfies constraints;
- is robust towards uncertainties.

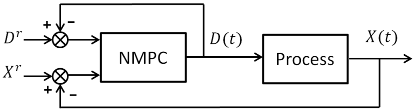


Figure: Overall NMPC strategy.

Algorithm

1. initialisation.
2. get the new state x;
3. solving the optimization problem over a finite moving horizon $\Downarrow$ (trust-region-reflective algorithm)
4. optimal input u ;
5. apply only the first value of the optimal sequence u;
6. $t \longleftarrow t+1$. Go to 2.

Discrete prediction model (Runge-Kutta scheme)

$$\begin{cases} x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} f(x(\tau), u_k, \theta) d\tau \\ y_k = Hx_k \end{cases} \tag{9}$$

where

- x_k: the discrete state vector at time k;
- y_k: the sampled measurement at time k;
- $u(k)$: control action (parametrized using a piecewise-constant approximation);

$$(u(\tau) = u(k),\ \tau \in [kT_s, (k+1)T_s])$$

- H: measurement matrix.

Discrete state trajectory

$$x_{k+1} = g(t_0, t_{k+1}, x_0, u_{t_0}^{t_k}, \theta) \tag{10}$$

with $u_{t_0}^{t_k}$ the control sequence from the initial time instant t_0 to the time instant t_k.

Mathematical formulation of NMPC

$$\tilde{u}_k^{k+N_p-1} = arg \min_{u_k^{k+N_p-1}} \Pi(u_k^{k+N_p-1}, \theta) \tag{11}$$

where

$$\Pi(u_k^{k+N_p-1}, \theta) = ||\hat{y}_{k+1}^{k+N_p} - \bar{y}_{k+1}^{k+N_p}||_P^2 + ||u_k^{k+N_p-1} - \bar{u}_k^{k+N_p-1}||_R^2 \tag{12}$$

with

$$\hat{y}_{k+1}^{k+N_p} = \begin{bmatrix} Hg(t_k, t_{k+1}, x_k, u_k, \theta) \\ Hg(t_k, t_{k+2}, x_k, u_k^{k+1}, \theta) \\ \vdots \\ Hg(t_k, t_{k+N_p}, x_k, u_k^{k+N_p-1}, \theta) \end{bmatrix}$$: the predicted output,

$\bar{y}_{k+1}^{k+N_p} = [y_{k+1}^r, \dots, y_{k+N_p}^r]^\top$: the setpoint values,

$u_k^{k+N_p-1} = [u_k, \dots, u_{k+N_p-1}]^\top$: the optimization variable,

$\bar{u}_k^{k+N_p-1} = [u_k^r, \dots, u_{k+N_p-1}^r]^\top$: the reference control sequence.

and the tuning diagonal matrices $P \geq 0$, $R > 0$.

In the case of uncertain parameters ($\Theta = [\theta^-, \theta^+]$):

Min-Max problem (RNMPC)

$$\tilde{u}_k^{k+N_p-1} = arg \min_{u_k^{k+N_p-1}} \max_{\hat{\theta}\in\Theta} \Pi(u_k^{k+N_p-1}, \hat{\theta}) \tag{13}$$

with

$$\Pi(u_k^{k+N_p-1}, \hat{\theta}) = ||\hat{y}_{k+1}^{k+N_p} - \bar{y}_{k+1}^{k+N_p}||_P^2 + ||u_k^{k+N_p-1} - \bar{u}_k^{k+N_p-1}||_R^2 \tag{14}$$

and

$$\hat{y}_{k+1}^{k+N_p} = \begin{bmatrix} Hg(t_k, t_{k+1}, x_k, u_k, \hat{\theta}) \\ Hg(t_k, t_{k+2}, x_k, u_k^{k+1}, \hat{\theta}) \\ \vdots \\ Hg(t_k, t_{k+N_p}, x_k, u_k^{k+N_p-1}, \hat{\theta}) \end{bmatrix} \tag{15}$$

The optimal control sequence $u_k^{k+N_p-1}$ is determined so that the maximum deviation for all trajectories over all possible data scenarii is minimized [Yu, 1998],[Kerrigan and Maciejowski, 2004].

Taylor series expansion (limited to the first-order)

g is linearized around $\bar{u}_k^{k+N_p-1}$ & $\theta_{nom} = \frac{\theta^+ + \theta^-}{2}$ for $j = \overline{1, N_p}$:

$$\begin{aligned} g(t_k, t_{k+j}, x_k, u_k^{k+j-1}, \hat{\theta}) \approx \quad & g(t_k, t_{k+j}, x_k, \bar{u}_k^{k+j-1}, \theta_{nom}) \\ & + \nabla g_u(t_{k+j-1})(u_k^{k+j-1} - \bar{u}_k^{k+j-1}) \\ & + \nabla g_\theta(t_{k+j})(\hat{\theta} - \theta_{nom}) \end{aligned} \tag{16}$$

Sensitivity functions

$$\nabla g_u(t_{k+j-1}) = \left. \frac{\partial g(t_k, t_{k+j}, x_k, u_k^{k+j-1}, \theta)}{\partial u_k^{k+j-1}} \right|_{\substack{u_k^{k+j-1} = \bar{u}_k^{k+j-1} \\ \theta = \theta_{nom}}} \tag{17}$$

$$\nabla g_\theta(t_{k+j}) = \left. \frac{\partial g(t_k, t_{k+j}, x_k, u_k^{k+j-1}, \theta)}{\partial \theta} \right|_{\substack{u_k^{k+j-1} = \bar{u}_k^{k+j-1} \\ \theta = \theta_{nom}}} \tag{18}$$

Dynamics of sensitivity functions with respect to θ and u

- Analytical derivation [Dochain, 2008]

$$\frac{d}{dt}(\nabla g_\theta) = \frac{\partial f(x,u,\theta_{nom})}{\partial x}\nabla g_\theta + \frac{\partial f(x,u,\theta)}{\partial \theta}\ |_{\theta=\theta_{nom}} \quad (19)$$

Initial conditions: $\nabla g_\theta(t_k) = 0_{3\times 6}$

- The computing of ∇g_u is done numerically by finite differences.

$$\hat{y}_{k+1}^{k+N_p} \approx \bar{G}_{nom,k+1}^{k+N_p} + \bar{G}_{u,k}^{k+N_p-1}(u_k^{k+N_p-1} - \bar{u}_k^{k+N_p-1}) + \bar{G}_{\theta,k+1}^{k+N_p}(\hat{\theta} - \theta_{nom}) \quad (20)$$

with
${\bar{G}_{nom,k+1}^{k+N_p}}^\top = [Hg_{nom}(t_{k+1}),\ldots,Hg_{nom}(t_{k+N_p})]$: predicted output for the nominal case.
${\bar{G}_{u,k+1}^{k+N_p}}^\top = [H\nabla_u g(t_{k+1}),\ldots,H\nabla_u g(t_{k+N_p})]$: Jacobian matrices related to the control sequence.
${\bar{G}_{\theta,k+1}^{k+N_p}}^\top = [H\nabla_\theta g(t_{k+1}),\ldots,H\nabla_\theta g(t_{k+N_p})]$: Jacobian matrices related to the parameters.

New cost function

$$\Pi(u_k^{k+N_p-1},\hat{\theta}) = ||u_k^{k+N_p-1} - \bar{u}_k^{k+N_p-1}||_R^2 + ||\bar{G}_{u,k}^{k+N_p-1}(u_k^{k+N_p-1} - \bar{u}_k^{k+N_p-1}) - (\bar{y}_{k+1}^{k+N_p} - \bar{G}_{nom,k+1}^{k+N_p} - \bar{G}_{\theta,k+1}^{k+N_p}(\hat{\theta} - \theta_{nom}))||_P^2 \quad (21)$$

Bounded parametric error (uncertain parameters are uncorrelated)

$$\hat{\theta} - \theta_{nom} = \gamma\delta\theta_{max}$$

with $\delta\theta_{max} = \frac{\theta^+ - \theta^-}{2}$ and $||\gamma|| \leq 1$

Approach

move from Min-Max optimization of $N_p \times N_\theta$ dimension to a scalar robust regularized optimization [Benattia et al., 2015].

Robust Regularized Least Squares Problem

$$\tilde{z} = arg\, \min_{z} \max_{E_b} ||z||_R^2 + ||Az - (b + C\gamma E_b)||_P^2 \quad (22)$$

$$\text{with} \begin{cases} z = u_k^{k+N_p-1} - \bar{u}_k^{k+N_p-1}, A = \bar{G}_{u,k}^{k+N_p-1} \\ b = \bar{y}_{k+1}^{k+N_p} - \bar{G}_{nom,k+1}^{k+N_p}, C = \bar{G}_{\theta,k+1}^{k+N_p}, E_b = -\delta\theta_{max} \end{cases} \quad (23)$$

Unique global minimum $z(\lambda^\circ)$

$$z(\lambda^\circ) = [R + A^\top P(\lambda^\circ) A]^{-1} A^\top P(\lambda^\circ) b \quad (24)$$

Modified weighting matrix $p(\lambda^\circ)$

$$p(\lambda^\circ) = P + PC(\lambda^\circ I - C^\top PC)^\dagger C^\top P \quad (25)$$

Nonnegative scalar parameter $\lambda^\circ (\lambda \in \mathbb{R}^+)$

$$\lambda^\circ = arg \min_{\lambda \geq ||C^\top PC||} ||z(\lambda)||_R^2 + \lambda || - E_b||^2 + ||Az(\lambda) - b||_{P(\lambda)}^2 \quad (26)$$

$$\text{with} \begin{cases} z(\lambda) = [R + A^\top P(\lambda) A]^{-1} A^\top P(\lambda) b \\ P(\lambda) = P + PC(\lambda I - C^\top PC)^\dagger C^\top P \end{cases}$$

Linearized Robust MPC (LRMPC)

$$\tilde{u}_k^{k+N_p-1} = \bar{u}_k^{k+N_p-1} + [R + \bar{G}_{u,k}^{k+N_p-1}{}^\top P(\lambda^\circ) \bar{G}_{u,k}^{k+N_p-1}]^{-1} [\bar{G}_{u,k}^{k+N_p-1}{}^\top P(\lambda^\circ)(\bar{y}_{k+1}^{k+N_p} - \bar{G}_{nom,k+1}^{k+N_p})]$$

Parameters uncertainties $\pm 20\%$ [Goffaux and Vande Wouwer, 2008]

- exploring the parameter subspace border $\{\theta^-, \theta^+\} \implies 2^{dim(\theta)}$ tests ;
- worst-case model mismatch: $\theta_{real} = [\rho_m^+ \; K_s^- \; \bar{\mu}^+ \; K_Q^- \; K_{sI}^- \; K_{iI}^+]$

- N_p=5;
- $P = I_{N_p}$ and $R = I_{N_p}$;

Compromise

- $T_s \searrow \implies$ first order Taylor series expansion is accurate as much as possible;
- $T_s \nearrow \implies$ computation burden due to the state estimator and/or determination online of the optimal trajectory.

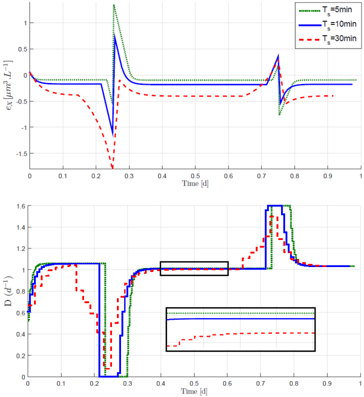


Figure: Biomass concentration tracking error and dilution rate evolution with time for LRMPC strategy.

Choice

$T_s = 10$ min $\implies$ linearization accuracy/computational burden $\surd$.

- $T_s = 10$ min;
- $P = I_{N_p}$ and $R = I_{N_p}$;

Compromise

- $N_p \searrow \Longrightarrow$ computation time and a sufficient vision of the system behaviour;
- $N_p \nearrow \Longrightarrow$ poor prediction accuracy.

Choice

$N_p = 5 \Longrightarrow$ good prediction $\surd$.

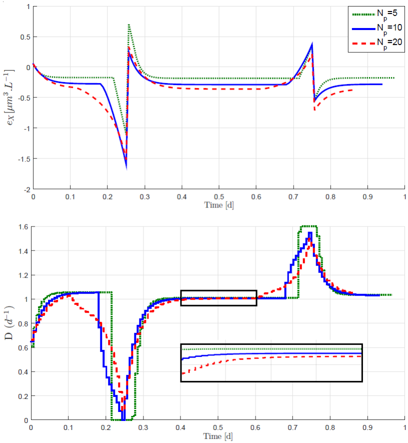


Figure: Biomass concentration tracking error evolution with time for LRMPC strategy.

- $y_k = X_k + e$ & $e \rightsquigarrow \aleph(0, 0.1)$;
- T_s=10 min and $N_p = 5$;
- $P = I_{N_p}$ and $R = I_{N_p}$;

- anticipation behavior to a setpoint change;
- dilution rate $\searrow \Longrightarrow$ cell concentration $\nearrow$ and vice versa(biological aspect $\surd$);
- both RNMPC and LRMPC have better performances than the classical NMPC;

LRMPC

approximation of the model through linearization
+ model mismatch
$\Downarrow$
static error

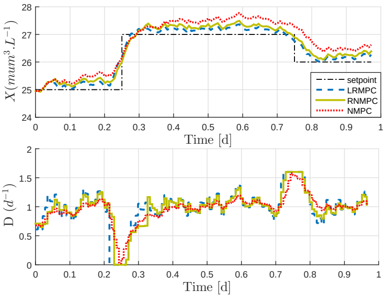


Figure: Biomass concentration and dilution rate evolution with time for NMPC, RNMPC and LRMPC strategies.

Table: Comparison of the proposed algorithms in terms of computation time at each sampling time (worst case).

Algo. \ Perf. indices	Computation time (s) min	mean	max
NMPC	$< 10^{-5}$	0.024	0.31
RNMPC	0.43	3.32	47.5
LRMPC	$< 10^{-5}$	0.016	0.1

Conclusion

- regulation of the biomass concentration to a desired value in a continuous photobioreactor;
- solving, online, NMPC problem using a finite parametrization of u;
- robust NMPC $\Longrightarrow$ min-max problem;
- first order Taylor series expansion $\Longrightarrow$ LRMPC law;
- comparison between the proposed strategy and the (R)NMPC laws;
- robustness of the proposed strategy w.r.t model parameters uncertainties;
- reducing the computational burden.

Perspectives

- hierarchical control strategy formed by two level controller:

$$\text{predictive} + \text{integral action}; \rightsquigarrow$$

- determination of conditions to ensure robust stability with bounded uncertainties including inequality constraints; $\rightsquigarrow$
- second order expansion to improve robustness; $\nearrow$
- estimation algorithm to reconstruct biomass concentration; $\nearrow$

Thank you for your attention!

Appendix A

The equilibrium is defined as follows:

$$\begin{cases} D = \bar{\mu}(1 - \frac{K_Q}{Q})\mu_I \\ S = \frac{\mu(Q,I)QK_s}{\rho_m - \mu(Q,I)Q} \\ (S_{in} - \frac{\mu(Q,I)QK_s}{\rho_m - \mu(Q,I)Q})\mu(Q,I) - \mu(Q,I)QX = 0 \end{cases} \tag{27}$$

Taking Q as an unknown variable, the following quadratic equation must be solved:

$$\bar{\mu}\mu_I X Q^2 - \{(S_{in} + K_s)\bar{\mu}\mu_I + (\rho_m + \bar{\mu}\mu_I K_Q)X\}Q + (\rho_m + \bar{\mu}\mu_I K_Q)S_{in} + \bar{\mu}\mu_I K_Q K_s = 0 \tag{28}$$

The discriminant of the equation (28) is given by:

$$\Delta = aX^2 - bX + c = (\sqrt{a}X - \sqrt{c})^2 + (2\sqrt{a}\sqrt{c} - b)X \tag{29}$$

where

$$\begin{cases} a = (\rho_m + \bar{\mu}\mu_I K_Q)^2 \\ b = -2\bar{\mu}\mu_I\{(K_s - S_{in})(\rho_m + \bar{\mu}\mu_I K_Q) - 2\bar{\mu}\mu_I K_Q K_s\} \\ c = \bar{\mu}^2\mu_I^2(S_{in} + K_s)^2 \end{cases} \tag{30}$$

we have that

$$2\sqrt{a}\sqrt{c} - b = 2(\rho_m + \bar{\mu}\mu_I K_Q)\bar{\mu}\mu_I(S_{in} + K_s) + 2\bar{\mu}\mu_I\{(K_s - S_{in})(\rho_m + \bar{\mu}\mu_I K_Q) - 2\bar{\mu}\mu_I K_Q K_s\}$$
$$2\sqrt{a}\sqrt{c} - b = 4\bar{\mu}\mu_I K_s \rho_m \tag{31}$$

Finally

$$\Delta = ((\rho_m + \bar{\mu}\mu_I K_Q)X - \bar{\mu}\mu_I(S_{in} + K_s))^2 + 4\bar{\mu}\mu_I K_s \rho_m X \tag{32}$$

Since the discriminant Δ is non negative, there are two real solutions :

$$Q^*_{1,2} = \frac{S_{in} + K_s}{2X} + \frac{1}{2}(\frac{\rho_m}{\bar{\mu}\mu_I} + K_Q) \mp \frac{\sqrt{\Delta}}{2\bar{\mu}\mu_I X} \tag{33}$$

Using (29), it comes:

$$\frac{\sqrt{\Delta}}{2\bar{\mu}\mu_I X} = \frac{1}{2}\sqrt{a}\sqrt{1 + \alpha\frac{1}{X} + \beta\frac{1}{X^2}} \tag{34}$$

with

$$\begin{cases} \alpha = -\frac{b}{a} \\ \beta = \frac{c}{a} \end{cases} \tag{35}$$

where

$$\sqrt{1 + \alpha\frac{1}{X} + \beta\frac{1}{X^2}} = \frac{1}{X}\sqrt{(X + \sqrt{\beta})^2 + (\alpha - 2\sqrt{\beta})X} \tag{36}$$

we have that

$$\alpha - 2\sqrt{\beta} = \frac{2\bar{\mu}\mu_I[(K_s - S_{in})(\rho_m + \bar{\mu}\mu_I K_Q) - 2\bar{\mu}\mu_I K_Q K_s]}{(\rho_m + \bar{\mu}\mu_I K_Q)^2} - \frac{2\bar{\mu}\mu_I(S_{in} + K_s)}{(\rho_m + \bar{\mu}\mu_I K_Q)}$$

$$= \frac{2\bar{\mu}\mu_I(K_s - S_{in})}{(\rho_m + \bar{\mu}\mu_I K_Q)} - \frac{4\bar{\mu}^2\mu_I^2 K_s K_Q}{(\rho_m + \bar{\mu}\mu_I K_Q)^2} - \frac{2\bar{\mu}\mu_I(S_{in} + K_s)}{(\rho_m + \bar{\mu}\mu_I K_Q)} \quad (37)$$

$$\alpha - 2\sqrt{\beta} = \frac{4\bar{\mu}\mu_I\rho_m K_s}{(\rho_m + \bar{\mu}\mu_I K_Q)^2}$$

Then

$$\sqrt{1 + \alpha\frac{1}{X} + \beta\frac{1}{X^2}} = \sqrt{\left(1 + \frac{\bar{\mu}\mu_I(S_{in} + K_s)}{X(\rho_m + \bar{\mu}\mu_I K_Q)}\right)^2 + \frac{4\bar{\mu}\mu_I\rho_m K_s}{X(\rho_m + \bar{\mu}\mu_I K_Q)^2}} > 1 \quad (38)$$

Finally

$$Q_2 = \frac{S_{in} + K_s}{2X} + \frac{1}{2}\left(\frac{\rho_m}{\bar{\mu}\mu_I} + K_Q\right)\{1 + \sqrt{1 + \alpha\frac{1}{X} + \beta\frac{1}{X^2}}\} \quad (39)$$

Appendix B

- **Robust regularized optimization problem**

$$z^{o} = arg\ \min_{z} \max_{\delta A, \delta b} ||z||_V^2 + ||(A + \delta A)z - (b + \delta b)||_W^2 \tag{40}$$

where $V > 0$ and $W \geq 0$ are Hermitian weighting matrices.

$$\text{Perturbation modeling}: \begin{cases} \delta A = C\Delta E_a \\ \delta b = C\Delta E_b \end{cases}, \quad C \neq 0 \tag{41}$$

Δ: arbitrary contraction ($||\Delta|| \leq 1$).

- **Constrained two player game problem**

$$\min_{z} \max_{||\kappa|| \leq \pi(z)} J(z, \kappa) \tag{42}$$

with

$$J(z, \kappa) = z^{\top} V z + (Az - b + C\kappa)^{\top} W (Az - b + C\kappa) \tag{43}$$

and

$$\kappa = \Delta(E_a z - E_b) \tag{44}$$

- **On the use of Lagrangean multiplier λ: (42) $\Longrightarrow$ an unconstrained problem**

$$\min_{z} \max_{\kappa,\lambda} J(z,\kappa,\lambda) \tag{45}$$

with

$$J(z,\kappa,\lambda) = z^\top Vz + (Az - b + C\kappa)^\top W(Az - b + C\kappa) - \lambda(||\kappa||^2 - \pi(z)^2) \tag{46}$$

solution of κ

1. First derivative: $\frac{\partial J(z,\kappa,\lambda)}{\partial \kappa} = 0$

$$C^\top W(Az-b) + C^\top WC\kappa + (Az-b)^\top WC + \kappa^\top C^\top WC - 2\lambda\kappa = 0$$
$$C^\top W(Az-b) + (Az-b)^\top WC + C^\top WC\kappa + \kappa^\top C^\top WC - 2\lambda\kappa = 0$$
$$C^\top W(Az-b) = (\lambda I - C^\top WC)\kappa$$

Then,

$$\kappa = (\lambda I - C^\top WC)^{-1} C^\top W(Az-b) \tag{47}$$

2. Second derivative: $\frac{\partial^2 J(z,\mu,\kappa,\lambda)}{\partial^2 \kappa} \geq 0$

$$C^\top WC + C^\top WC - 2\lambda \geq 0$$

Then,

$$\lambda \geq ||C^\top WC||$$

Using the analytic solution (47) in the cost function (46), we have:

$$J(z,\lambda) = \quad z^\top Vz + (Az-b)^\top W(Az-b) + (Az-b)^\top WC\kappa + \kappa^\top C^\top W(Az-b) + \kappa^\top C^\top WC\kappa - \lambda\kappa^\top\kappa + \lambda\pi(z)^2 \tag{48}$$

Then,

$$J(z,\lambda) = z^\top Vz + (Az-b)^\top W(\lambda)(Az-b) + \lambda\pi(z)^2 \tag{49}$$

with

$$W(\lambda) = W + WC(\lambda I - C^\top WC)^{-1}C^\top W \tag{50}$$

Finally, the optimization problem (45) becomes:

$$\underset{z}{min} \underset{\lambda \geq ||C^\top WC||}{min} J(z,\lambda) \Longleftrightarrow \underset{\lambda \geq ||C^\top WC||}{min} \underset{z}{min} J(z,\lambda) \tag{51}$$

Analytic solution of z

$$\frac{\partial J(z,\lambda)}{\partial z} = 0 \tag{52}$$

$$Vz + z^\top V + A^\top W(\lambda)(Az-b) + (Az-b)^\top W(\lambda)A + \lambda\nabla\pi(z)^2 = 0 \tag{53}$$

$$(V + A^\top W(\lambda)A)z + \frac{1}{2}\lambda\nabla\pi(z)^2 = A^\top W(\lambda)b \tag{54}$$

We have that:

$$\pi(z) = ||E_a z - E_b|| \tag{55}$$

Then,

$$\nabla \pi(z)^2 = 2E_a^\top (E_a z - E_b) \tag{56}$$

Using (56) in the analytic solution (54), it comes:

$$(V + A^\top W(\lambda)A)z + \lambda E_a^\top E_a z - \lambda E_a^\top E_b = A^\top W(\lambda)b$$

$$(V + A^\top W(\lambda)A + \lambda E_a^\top E_a)z = A^\top W(\lambda)b + \lambda E_a^\top E_b$$

Then,

$$z(\lambda) = E(\lambda)^{-1}B(\lambda) \tag{57}$$

with

$$\begin{cases} E(\lambda) = V(\lambda) + A^\top W(\lambda)A \\ B(\lambda) = A^\top W(\lambda)b + \lambda E_a^\top E_b \end{cases} \tag{58}$$

and

$$\begin{cases} W(\lambda^o) = W + WC(\lambda^o I - C^\top WC)^\dagger C^\top W \\ V(\lambda^o) = V + \lambda^o E_a^\top E_a \end{cases} \tag{59}$$

Replacing the analytic solution (57) in the cost function (49):

$$
\begin{aligned}
J(\lambda) &= z^\top R z + (Az-b)^\top W(\lambda)(Az-b) + \lambda(E_a z - E_b)^\top (E_a z - E_b) \\
&= z^\top R z + b^\top W(\lambda) b + z^\top A^\top W(\lambda) A z - b^\top W(\lambda) A z - z^\top A^\top W(\lambda) b \\
&\quad + \lambda z^\top E_a^\top E_a z + \lambda E_b^\top E_b - \lambda z^\top E_a^\top E_b - \lambda E_b^\top E_a z \\
&= \lambda E_b^\top E_b + b^\top W(\lambda) b + z^\top (R + A^\top W(\lambda) A + \lambda E_a^\top E_a) z - (b^\top W(\lambda) A + \lambda E_b^\top E_a) z \\
&\quad - z^\top (A^\top W(\lambda) b + \lambda E_a^\top E_b) \\
&= \lambda E_b^\top E_b + b^\top W(\lambda) b + z^\top (R + A^\top W(\lambda) A + \lambda E_a^\top E_a) z \\
&\quad - (b^\top W(\lambda) A + \lambda E_b^\top E_a) z - z^\top (A^\top W(\lambda) b + \lambda E_a^\top E_b) \\
&= \lambda E_b^\top E_b + b^\top W(\lambda) b + z^\top E(\lambda) z - B(\lambda)^\top z - z^\top B(\lambda) \\
J(\lambda) &= \lambda E_b^\top E_b - B(\lambda)^\top E(\lambda)^{-1} B(\lambda)
\end{aligned}
$$

The scalar λ is computed from the following minimization problem:

$$
\lambda^o = arg \min_{\lambda \geq ||H^\top W H||} J(\lambda) \tag{60}
$$

with

$$
J(\lambda) = \lambda ||E_a z(\lambda) - E_b||^2 + ||z(\lambda)||_R^2 + ||Az(\lambda) - b||_{W(\lambda)}^2 \tag{61}
$$

Finally, the unique global minimum z^o given by:

$$
z^o = E(\lambda^o)^{-1} B(\lambda^o) \tag{62}
$$

Abdollahi, J. and Dubljevic, S. (2012).
Lipid production optimization and optimal control of heterotrophic microalgae fed-batch bioreactor.
Chem. Eng. Sci, 84:619–âĹŠ627.

Benattia, S. E., Tebbani, S., and Dumur, D. (2015).
Robust nonlinear model predictive control for regulation of microalgae culture in a continuous photobioreactor.
Proc. of the 1st Conference on Modelling Identification and Control of Nonlinear Systems.
St. Petersburg,Russia.

Bernard, O. and Gouzé, J.-L. (1995).
Transient behavior of biological loop models, with application to the droop model.
Mathematical Biosciences, 127.

Camacho, E. F. and Bordons, C. (2004).
Model Predictive Control.
Springer London.

Dochain, D. (2008).
Automatic control of bioprocesses.
ISTE Ltd and John Wiley & Sons, Inc.

Dochain, D. and Perrier, M. (2004).
Adaptive backstepping nonlinear control of bioprocesses.
7th International Symposium, Advanced control of chemical processes, pages 77–82.

Goffaux, G. and Vande Wouwer, A. (2008).
Design of a robust nonlinear receding-horizon observer-Application to a biological system.
Proc. of the 17th IFAC World Congress, pages 15553–15558.

Ifrim, G. A., Titica, M., Barbu, M., Boillereaux, L., Cogne, G., Caraman, S., and Legrand, J. (2013).
Multivariable feedback linearizing control of chlamydomonas reinhardtii photoautotrophic growth process in a torus photobioreactor.
Chem. Eng. J., 218:191–203.

Jenzsch, M., Simutis, R., and Luebbert, A. (2006).
Generic model control of the specific growth rate in recombinant escherichia coli cultivations.
J. of Biotechnol., 122:483–493.

Kerrigan, E. C. and Maciejowski, J. (2004).
Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution.
Int. J. Robust Nonlinear Control, 14:395–413.

Mailleret, L., Bernard, O., and Steyer, J. P. (2004).
Nonlinear adaptive control for bioreactors with unknown kinetics.
Automatica, 40:1379–âĹŠ1385.

Masci, P., Grognard, F., and Bernard, O. (2010).
Microalgal biomass surface productivity optimization based on a photobioreactor model.
Proc. of the 11th CAB Conference.

Munoz-Tamayo, R., Martinon, P., Bougaran, G., Mairet, F., and Bernard, O. (2014).
Getting the most out of it: optimal experiments for parameter estimation of microalgae growth models.
J. of Process Control, 24(6):991–1001.

Sayed, A. H., Nascimento, V. H., and Cipparrone, F. A. M. (2002).
A regularized robust design criterion for uncertain data.
SIAM J. MAT. ANAL. APPL., 32:4:1120–1142.

Selisteanu, D., Petre, E., and Rasvan, V. (2007).
Sliding mode and adaptive sliding mode control of a class of nonlinear bioprocesses.
Inter. J. of Adap. Contr. and Sig. Proc., 21:795–âĹŠ822.

Tebbani, S., Lopes, F., and Becerra-Celis, G. (2015).
Nonlinear control of continuous cultures of *Porphyridium purpureum* in a photobioreactor.
Chem. Eng. Sci., 123:207–219.

Yu, G. (1998).
Min-max optimization of several classical discrete optimization problems.
J. of Opt. Theory and Appl., 98(1):221–242.