Timed Automata Abstraction of Switched Dynamical Systems using Control Funnels

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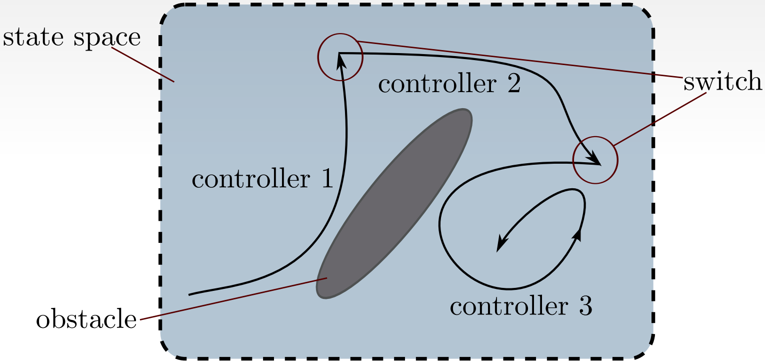
General problem and objective

- A controlled dynamical system ($\dot{x} = f(x, u, t)$) with a fixed finite set of controllers.



- Objective : create an abstraction and use formal methods for control design.

CONTROL FUNNELS

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t)$$

A control funnel or funnel is a trajectory $\mathcal{F}(t)$ of **sets in the state space** such that, for any trajectory $\boldsymbol{x}(t)$ of the dynamical system :

$$\forall t_0 \in \mathbb{R},\ \boldsymbol{x}(t_0) \in \mathcal{F}(t_0) \Rightarrow \forall t \geq t_0,\ x(t) \in \mathcal{F}(t)$$

EXAMPLES

$$\dot{x} = f(x, t)$$

For a continuously differentiable function $V(x, t)$, we define $\Omega_t = \{x | V(x, t) \leq 1\}$.

If for any $t \in \mathbb{R}$, and any $x \in \Omega_t$, $\frac{d}{dt}(V(x(t), t)) < 0$, i.e. :

$$\frac{\partial V}{\partial x}(x, t) f(x, t) + \frac{\partial V}{\partial t}(x, t) < 0,$$

then the function $\mathcal{F}(t) = \Omega_t$ is a funnel : for any $t_0 \in \mathbb{R}$,

$$x(t_0) \in \Omega_{t_0} \Rightarrow \forall t \geq t_0,\ x(t) \in \Omega_t.$$

Examples

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EXAMPLES

$$\dot{x} = x_{target} - x$$

For any set $W \subset \mathbb{R}^d$, the function $\mathcal{F}(t) = \{x_{target} + \exp(-t)w \mid w \in W\}$ is a funnel.

Indeed, if $x(t_0) = x_{target} + \exp(-t_0)w$, then for $t \geq t_0$,

$$x(t) = x_{target} + \exp(-t + t_0)(x(t_0) - x_{target}),$$

$$x(t) = x_{target} + \exp(-t)w,$$

and thus $x(t) \in \mathcal{F}(t)$.

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Examples

$$\dot{\boldsymbol{x}} = \boldsymbol{x}_{target} - \boldsymbol{x}$$

For any set $W \subset \mathbb{R}^d$, the function
$\mathcal{F}(t) = \{\boldsymbol{x}_{target} + \exp(-t)\boldsymbol{w} \mid \boldsymbol{w} \in W\}$ is a funnel.

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$$\boldsymbol{x}(t) = \boldsymbol{x}_{target} + \exp(-t)\boldsymbol{w},$$

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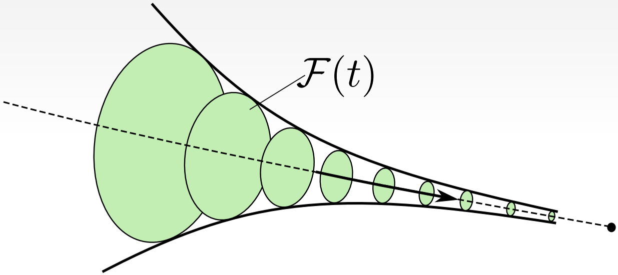
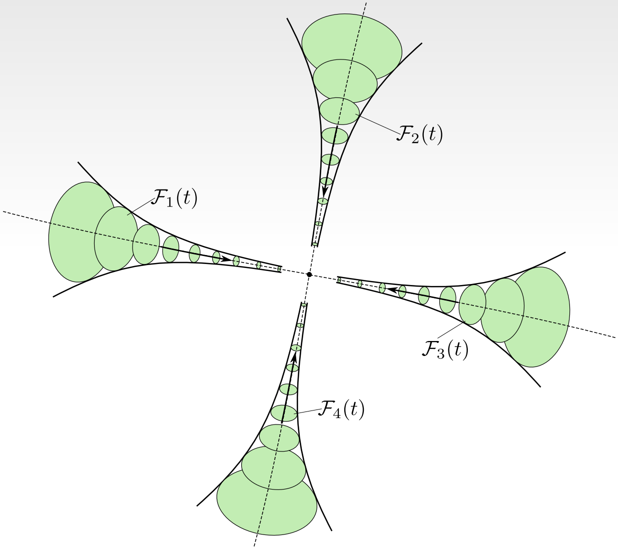
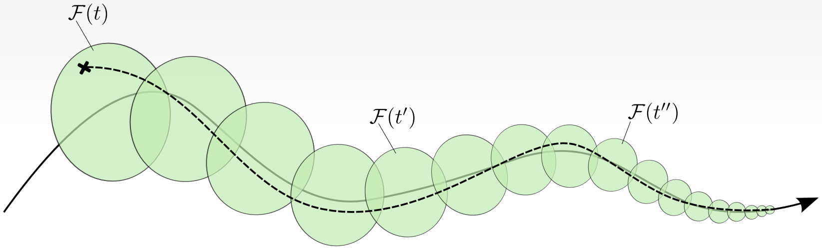
Examples



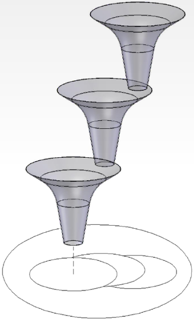
Examples



Examples

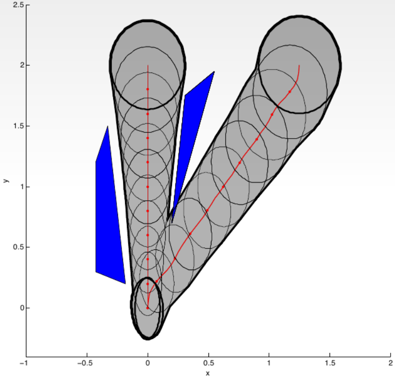


Motion planning with funnels



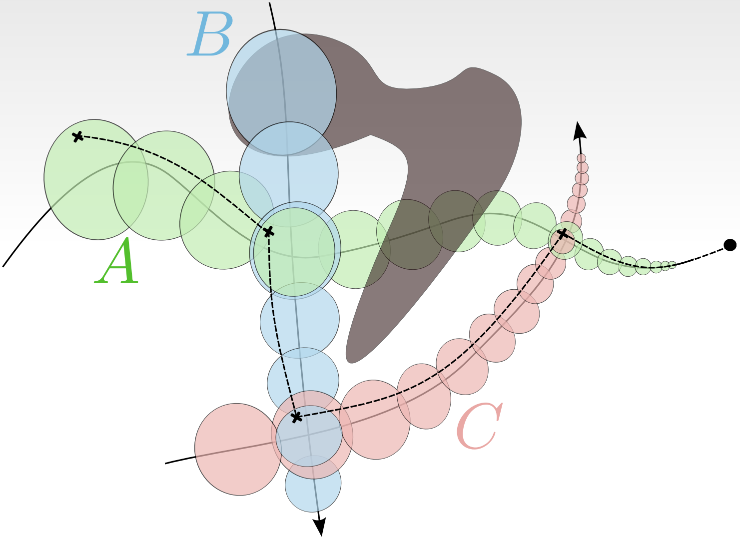
[Mason 1985], [Burridge et al. 1999], [Tedrake 2009]

Motion planning with funnels



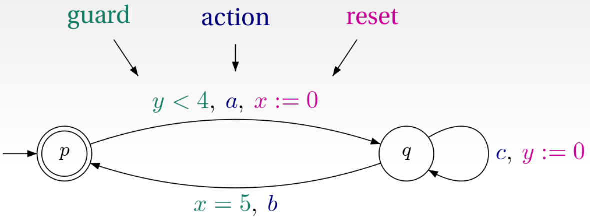
[Tobenkin, Manchester & Tedrake 2014]

Motion planning with funnels



Timed automata [Alur & Dill 1994]

x, y: clocks



TIMED AUTOMATA [ALUR & DILL 1994]

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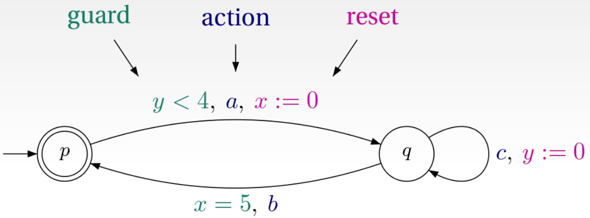


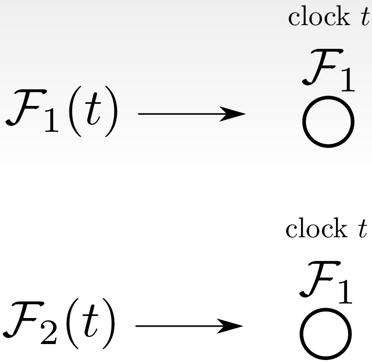
	p	$\xrightarrow[3.2]{a}$	q	$\xrightarrow[5.1]{c}$	q	$\xrightarrow[8.2]{b}$	p	...
value of x	0		0		1.9		5	...
value of y	0		3.2		0		3.1	...

➜ timed word $(a, 3.2)(c, 5.1)(b, 8.2)\ldots$

TIMED AUTOMATA [ALUR & DILL 1994]

- Reachability in timed automata is PSPACE-complete. Some implementations are efficient in practice (e.g. UPPAAL).
- Reachability and safety games on timed automata are decidable and EXPTIME-complete (and memoryless strategies are sufficient).

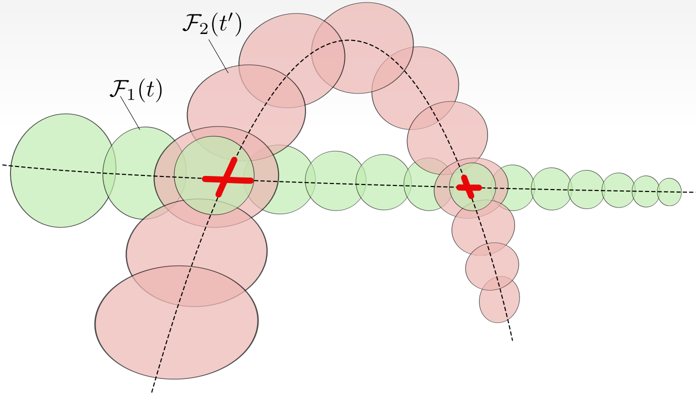
Abstraction of control funnels with timed automata



- Transitions : $\mathcal{F}_1(t) \rightarrow \mathcal{F}_2(t')$?

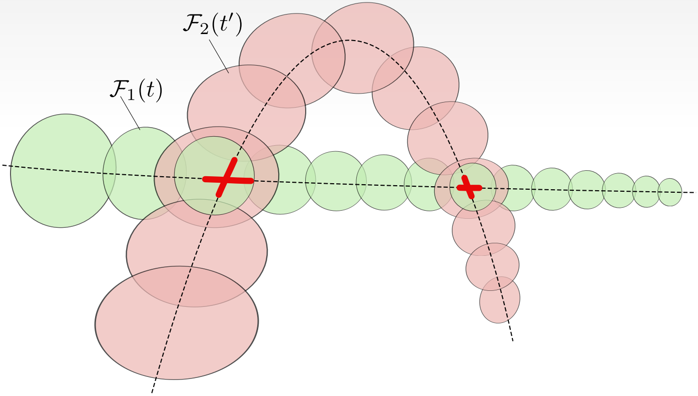
Abstraction of control funnels

$$(a' < t' < b' \wedge a < t < b) \vee (c' < t' < d' \wedge c < t < d)$$

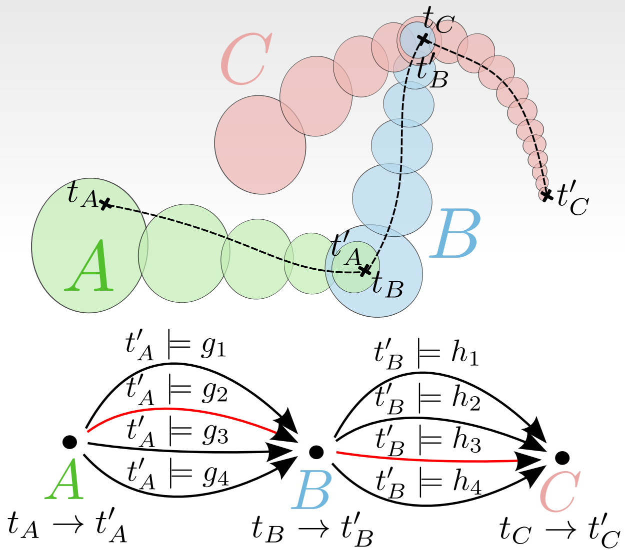


Abstraction of control funnels

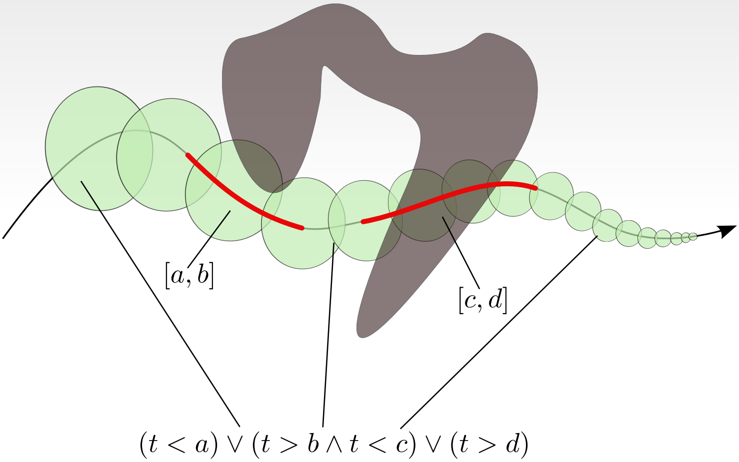
$$(\mathbf{t}' = \mathbf{a}' + \frac{\mathbf{k}}{\mathbf{N}}(\mathbf{b}' - \mathbf{a}') \wedge a < t < b)\ldots$$
$$\vee(\mathbf{t}' = \mathbf{c}' + \frac{\mathbf{q}}{\mathbf{N}}(\mathbf{d}' - \mathbf{c}') \wedge c < t < d)$$

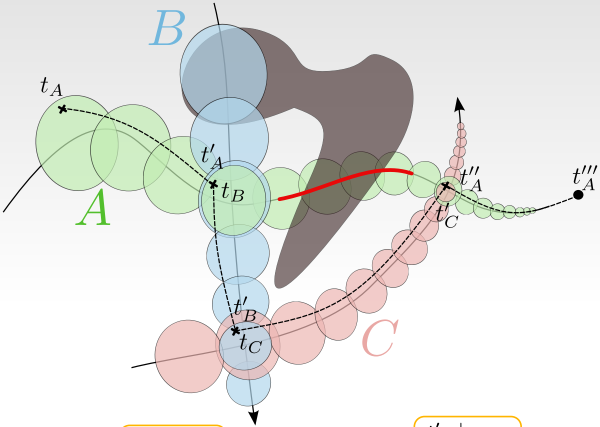


Abstraction of control funnels

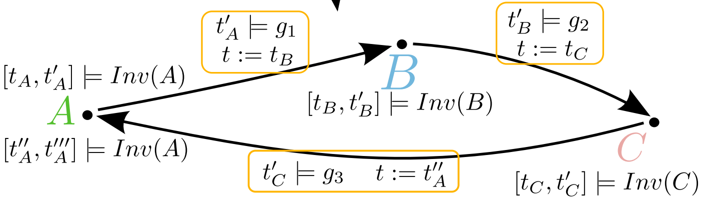


OBSTACLES → INVARIANTS ON CLOCKS



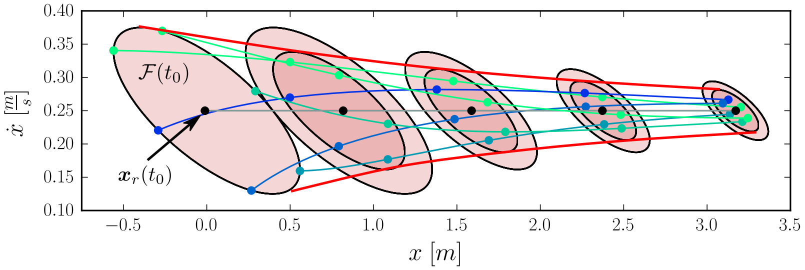


B
t_A
t'_A
t_B
A
t''_A
t'''_A
t'_C
t'_B
t_C
C

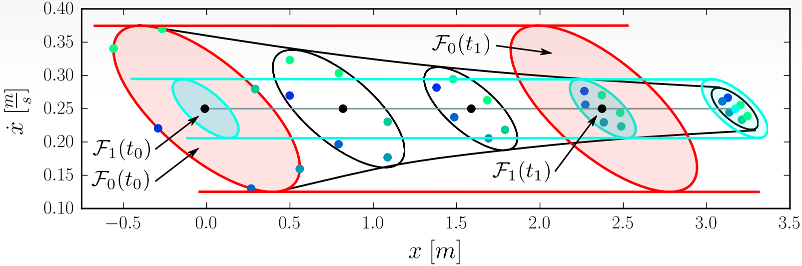


$t'_A \models g_1$
$t := t_B$
$t'_B \models g_2$
$t := t_C$
$[t_A, t'_A] \models Inv(A)$
B
$[t_B, t'_B] \models Inv(B)$
A
$[t''_A, t'''_A] \models Inv(A)$
C
$t'_C \models g_3$ $t := t''_A$
$[t_C, t'_C] \models Inv(C)$

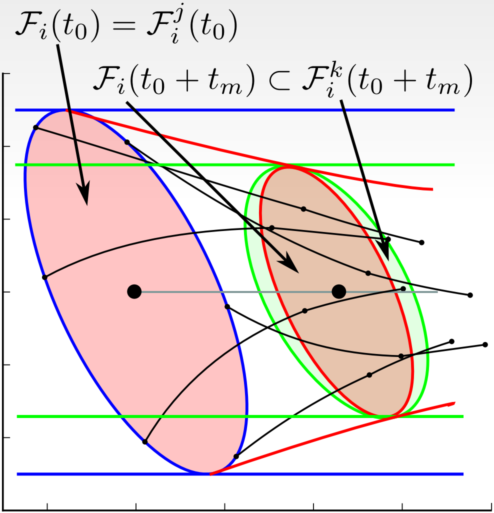
FIXED-SIZE CONTROL FUNNELS AND A NEW CLOCK



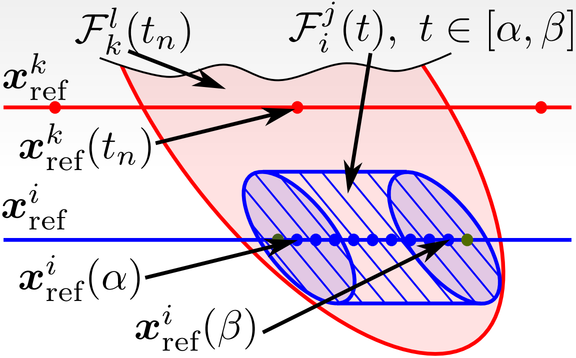
FIXED-SIZE CONTROL FUNNELS AND A NEW CLOCK



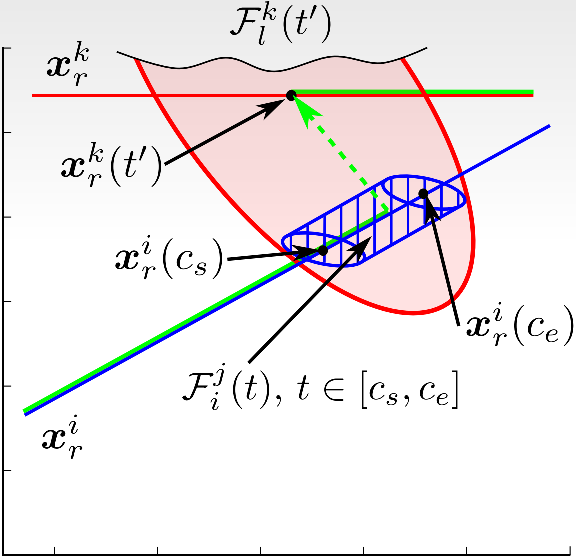
Fixed-size control funnels and a new clock



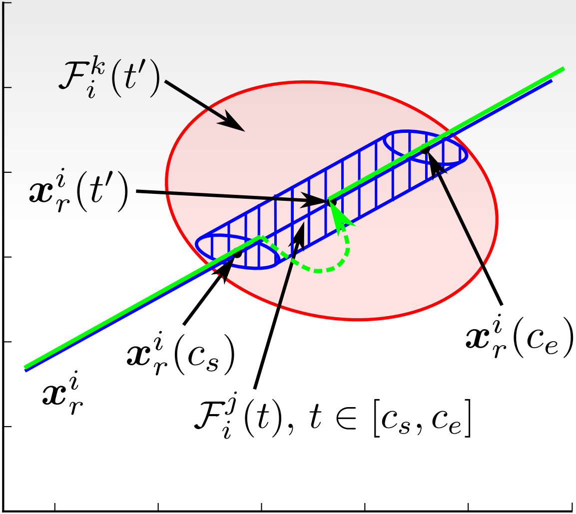
INCLUSION TESTS



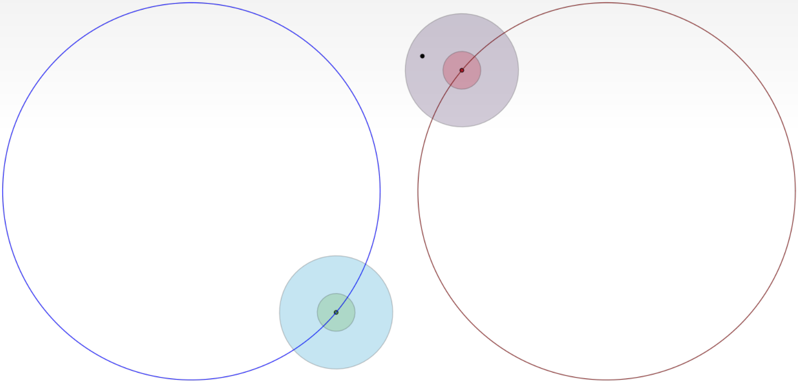
Reference trajectory switches



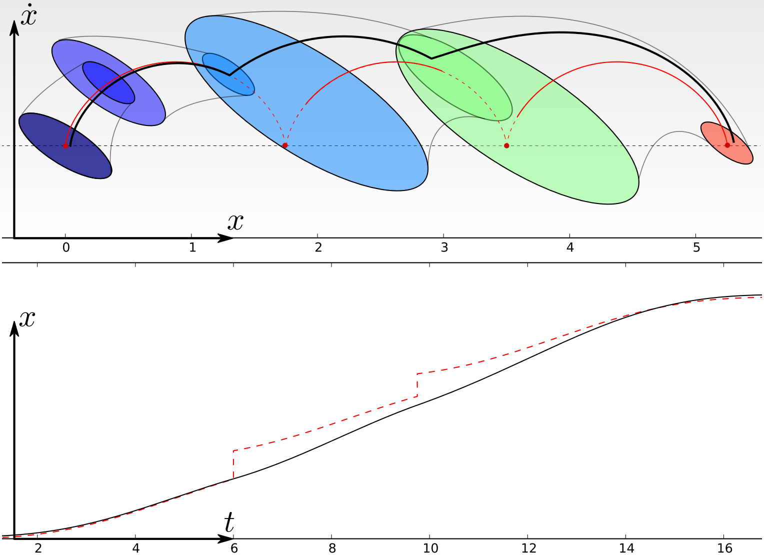
LOCAL ACCELERATIONS



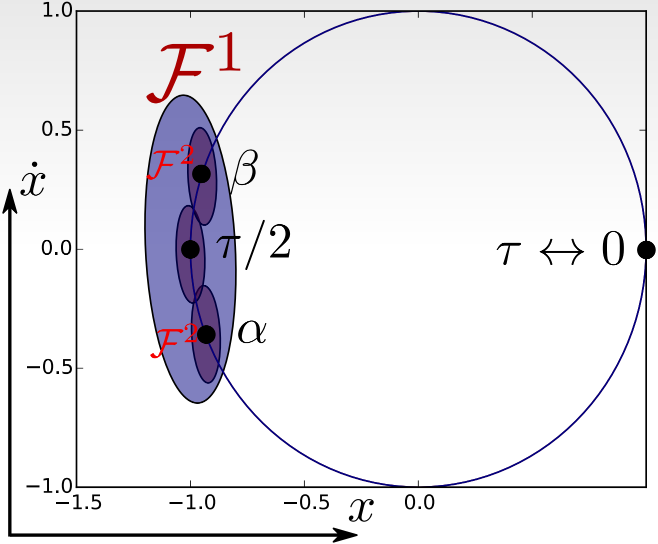
Simple 1D example with fixed-size control funnels



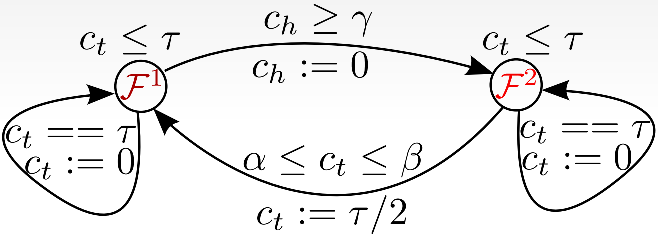
LQR funnels for linear systems



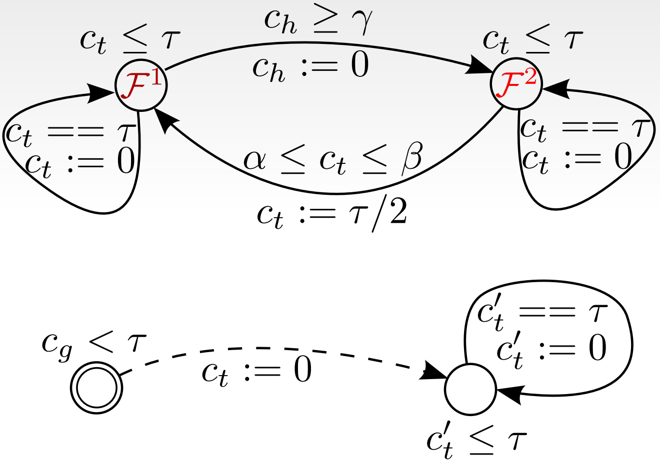
A synchronization game



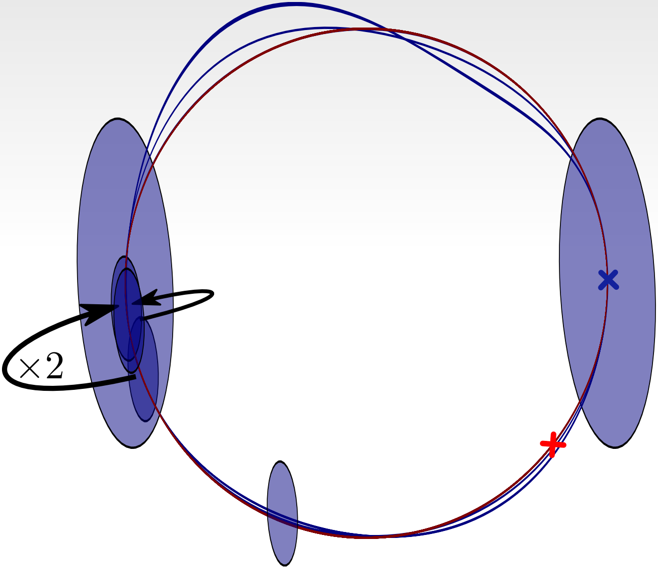
A SYNCHRONIZATION GAME



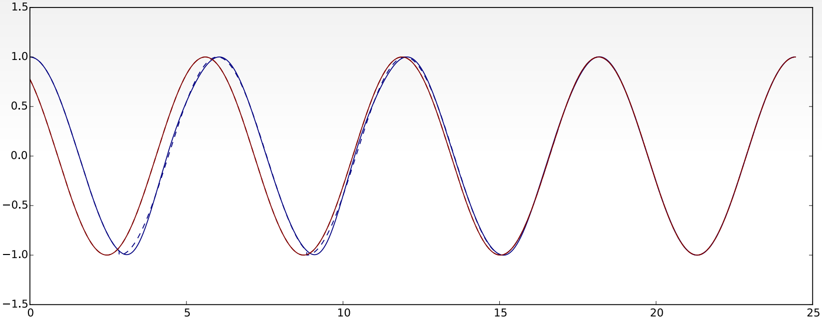
A synchronization game



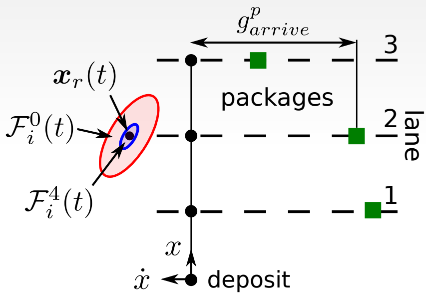
A synchronization game

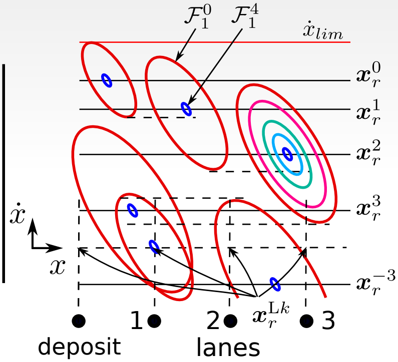


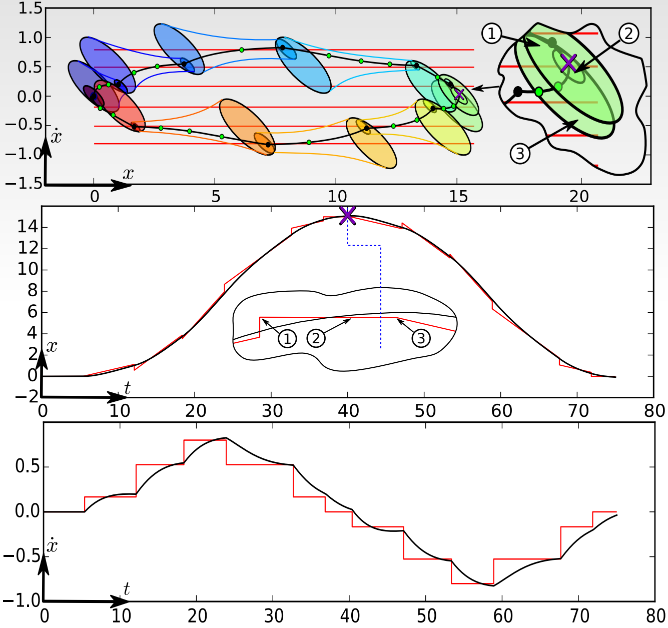
A synchronization game

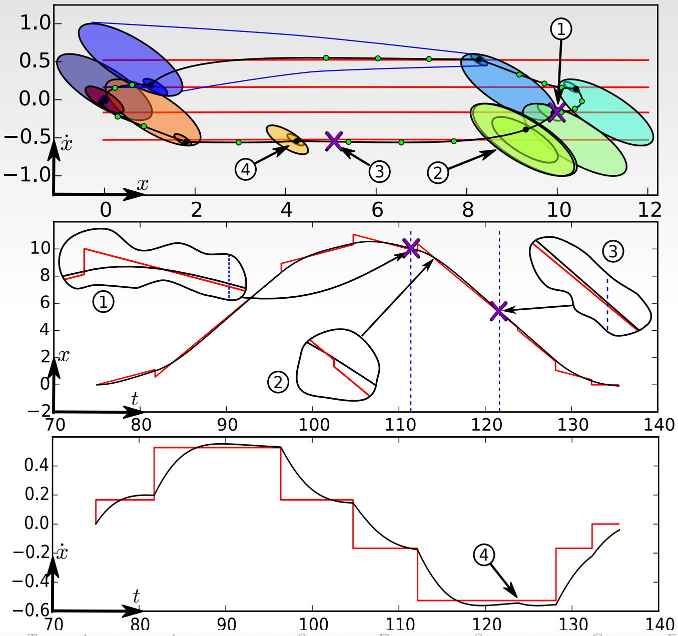


A 1D PICK-AND-PLACE PROBLEM

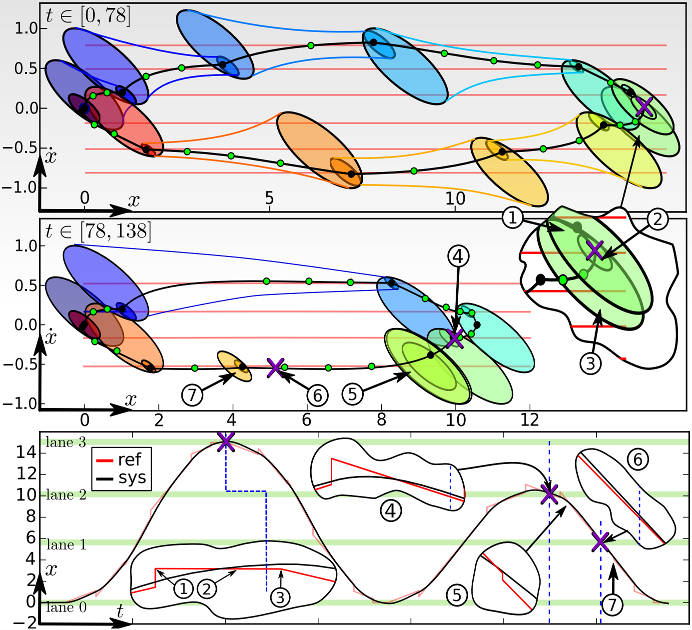





1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
$\dot{x}$
x
0
5
10
15
20
1
2
3
14
12
10
8
6
4
2
0
−2
x
t
0
10
20
30
40
50
60
70
80
1
2
3
0.5
0.0
−0.5
−1.0
$\dot{x}$
t
0
10
20
30
40
50
60
70
80



1.0
0.5
0.0
−0.5
−1.0
$\dot{x}$
x
0
2
4
6
8
10
12
①
②
③
④
10
8
6
4
2
0
−2
x
t
70
80
90
100
110
120
130
140
0.4
0.2
0.0
−0.2
−0.4
−0.6
$\dot{x}$
t


$t \in [0, 78]$
$t \in [78, 138]$
ref
sys
lane 3
lane 2
lane 1
lane 0

Future work

- Main goal : scale up to more practical problems.
- How ? By combining formal methods and numerical methods (optimization, learning, etc.).