Any continuous PWA Control law can be obtained by linear parametric optimization corresponding to a MPC with 2 prediction steps

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Outline

Motivation

Basic nomenclature

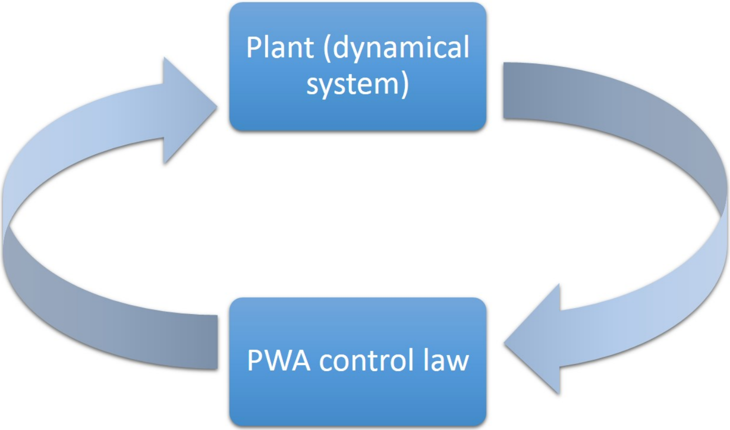
Convex liftings
- Existence conditions
- Constructive algorithm
- Nonconvexly liftable partitions

Solution to IPL/QP

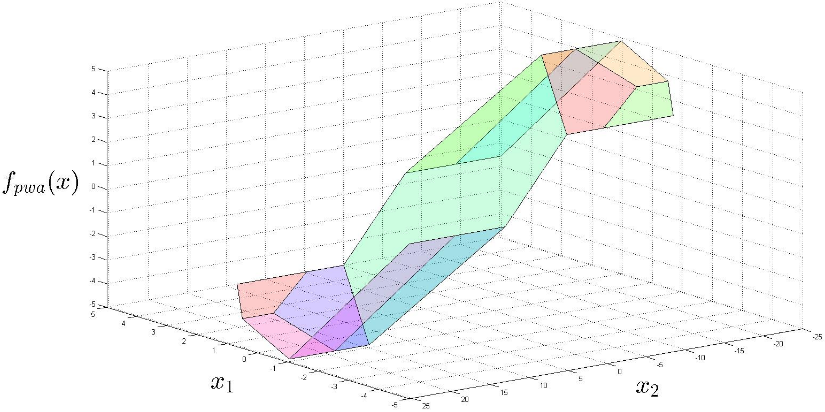
Related results
- Applications to PL/QP
- Applications to linear MPC designs

Numerical examples

A feedback loop with a PWA control law



PWA control law



Linear MPC

Given a LTI system: $\boxed{x_{k+1} = Ax_k + Bu_k}$

Minimize:

$$\boxed{J(U, x_k) = \left[\begin{array}{l} \sum_{i=0}^{N-1} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} + x_{k+N|k}^T P x_{k+N|k} \\ \text{or} \\ \sum_{i=0}^{N-1} \|Qx_{k+i|k}\|_{1/\infty} + \|Ru_{k+i|k}\|_{1/\infty} + \|Px_{k+N|k}\|_{1/\infty} \end{array}\right.}$$

s.t. $\boxed{\begin{array}{l} x_{k+i|k} \in \mathbb{X},\ u_{k+i|k} \in \mathbb{U} \text{ for } i = 0 \ldots N-1 \\ x_{k+N|k} \in \mathbb{X}_T \end{array}}$

MPC - Explicit solution

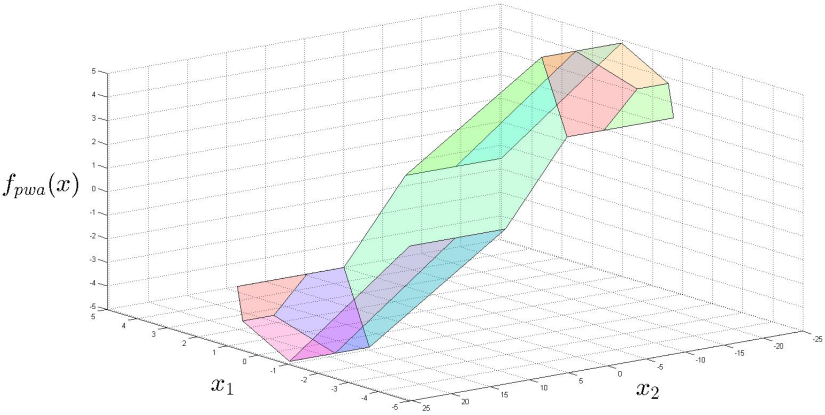
Parametric optimization problem

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} \mathbf{u}^T H \mathbf{u} + (x^T F + C)\mathbf{u}$$
$$\text{s.t. } G\mathbf{u} \leq Ex + W$$

Explicit solution

$$\begin{aligned} \mathbf{u}^* = f_{pwa} : \bigcup_{i=1}^{N} \mathcal{X}_i \subset \mathbb{R}^{d_x} &\longrightarrow \mathbb{R}^{d_u} \\ x \quad &\longmapsto f_i x + g_i \text{ for } x \in \mathcal{X}_i \end{aligned}$$

Explicit solution



Major inconveniences of explicit solution

With respect to MPC design:

- numerous regions to store
- difficult point-location problem

Our objective: revert to implicit MPC but use the minimal control horizon.

More general

With respect to the inverse optimality studies:

R. E. Kalman. When is a linear control system optimal? Trans. ASME J. Basic Engr, 86D:51–60, 1964.

Our objective: find "when a PWA (control) system is optimal".

Problem statement

Given polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^{d_x}$ and a continuous piecewise affine function $f_{pwa} : \mathcal{X} \to \mathbb{R}^{d_u}$, find $J(x, u, z), H_x, H_u, H_z, K$ such that

$$\begin{cases} f_{pwa}(x) = Proj_{\mathbb{R}^{d_u}} \arg \min\limits_{\begin{bmatrix} u^T & z \end{bmatrix}^T} J(x, u, z), \\ \text{s.t.} \quad H_u u + H_x x + H_z z \leq K. \end{cases}$$

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Polyhedral partition

A collection of $N \in \mathbb{N}_+$ full-dimensional polytopes $\mathcal{X}_i \subset \mathbb{R}^d$ is called a *polyhedral partition* if:

1. $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$ is a compact set in $\mathbb{R}^d$.
2. $\text{int}(\mathcal{X}_i) \bigcap \text{int}(\mathcal{X}_j) = \emptyset$ with $i \neq j$, $(i, j) \in \mathcal{I}_N^2$,

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours if $(i, j) \in \mathcal{I}_N^2$, $i \neq j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = d - 1$.

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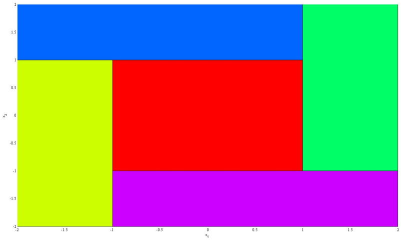

Cell complex

A *cell complex* $\mathcal{C}$ is defined as a set of polytopes provided:

- every face of a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$.
- the intersection of any two members of $\mathcal{C}$ is a face of each of them.

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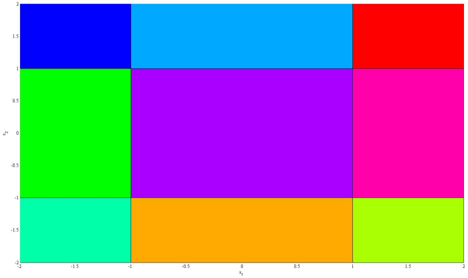

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Convex liftings

Given a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^d$, a *piecewise affine lifting* is described by a function:

$$
\begin{aligned}
z : \mathcal{X} &\to \mathbb{R} \\
x &\mapsto z(x) = A_i^T x + a_i \quad \text{for} \quad x \in \mathcal{X}_i,
\end{aligned}
\tag{1}
$$

where $A_i \in \mathbb{R}^d$ and $a_i \in \mathbb{R}$.

Convex liftings

Given a polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^d$, a *piecewise affine lifting* is described by a function:

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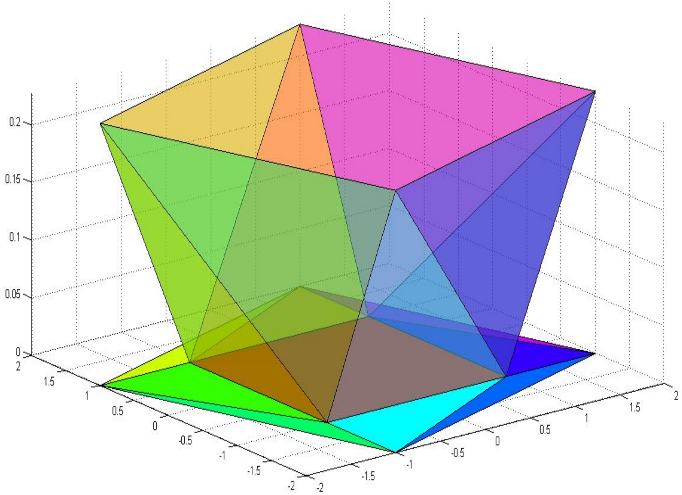
Given a cell complex $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^d$, a piecewise affine lifting $z(x) = A_i^T x + a_i \; \forall x \in \mathcal{X}_i$, is called *convex piecewise affine lifting* if the following conditions hold true:

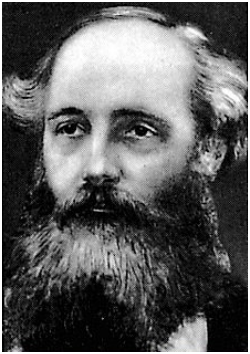
- $z(x)$ is continuous over $\mathcal{X}$,
- for each $i \in \mathcal{I}_N$, $z(x) > A_j^T x + a_j$ for all $x \in \mathcal{X}_i \backslash \mathcal{X}_j$ and all $j \neq i$, $j \in \mathcal{I}_N$.

Convex liftings



History

- James Clerk Maxwell(1831-1879), Scottish, mathematician, physicist
- He is the first one putting forward the notion of *reciprocal diagram* of a *cell complex* which is isomorphic to *convex lifting* in the plane $\mathbb{R}^2$.
- Links the geometrical problem to the notion of $k-$stress



$k-$stress

$n(F, C)$ inward unit normal vector to C at its facet F
$s(C) \in \mathbb{R}$: stress on the face C

A real-valued function $s(\cdot)$ defined on the $(k-1)-$faces of a polyhedral cell complex $K \subset \mathbb{R}^k$ is called a $k-$stress if at each internal $(k-2)-$face F of K:

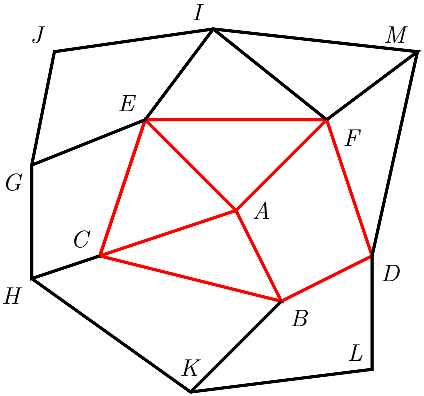
$$\sum_{C | F \subset C} s(C) n(F, C) = 0, \tag{2}$$

where this sum ranges over all $(k-1)-$faces in the star of F (the $(k-1)-$faces such that F is their common facet). The quantities $s(C)$ are the coefficients of the $k-$stresses, are called a *tension* if the sign is strictly positive, and a *compression* if the sign is strictly negative.

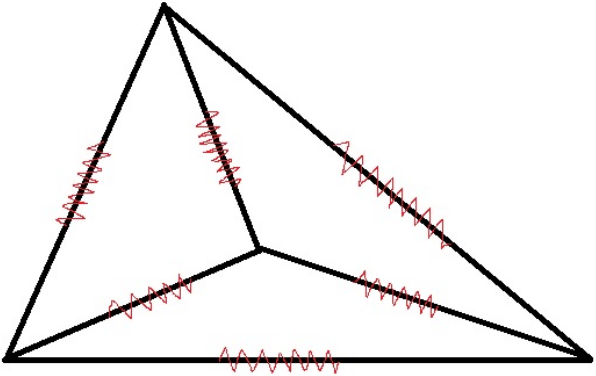
$k-$stress

$$s(AE)n_{AE} + s(AF)n_{AF} + s(AB)n_{AB} + s(AC)n_{AC} = 0,$$

$$n_{AE} = \frac{\overrightarrow{AE}}{AE}, n_{AF} = \frac{\overrightarrow{AF}}{AF}, n_{AB} = \frac{\overrightarrow{AB}}{AB}, n_{AC} = \frac{\overrightarrow{AC}}{AC}$$



Mechanical implication

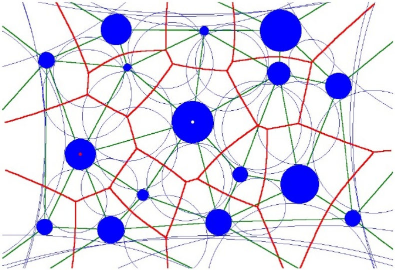


Existence conditions

Computational geometry

- it admits a strictly positive $d-$ stress,
- it is an additively weighted Dirichlet-Voronoi diagram,
- it is an additively weighted Delaunay diagram,
- it is the section of a $(d+1)-$dimensional Dirichlet-Voronoi diagram,
- it has a dual partition.

Gueorgui Feodossievitch Voronoi (1868-1908)



$\boxed{\textbf{We need an algorithm to construct convex liftings!}}$

A construction of convex liftings

Input: A given cell complex $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^d$.
Output: $(A_i,\ a_i), \forall i \in \mathcal{I}_N$.

1: Register all couples of neighboring regions in $\mathcal{X}$.
2: For each couple $(i, j) \in \mathcal{I}_N^2$ such that $(\mathcal{X}_i, \mathcal{X}_j)$ are neighbors:

$$A_i^T v + a_i = A_j^T v + a_j, \forall v \in \mathrm{vert}(\mathcal{X}_i \cap \mathcal{X}_j).$$
$$A_i^T u + a_i > A_j^T u + a_j, \forall u \in \mathrm{vert}\mathcal{X}_i,\ u \notin \mathrm{vert}(\mathcal{X}_i \cap \mathcal{X}_j).$$

3: $$f = \sum_{i=1}^N A_i^T A_i.$$
4:
$$\widetilde{\mathcal{X}} = conv\left\{ \begin{array}{c} \begin{bmatrix} v \\ z(v) \end{bmatrix} \in \mathbb{R}^{d+1} \mid v \in \bigcup_{i \in \mathcal{I}_N} \mathrm{vert}\mathcal{X}_i, \\ z(v) = A_i^T v + a_i \quad \text{if} \quad v \in \mathcal{X}_i \end{array} \right\}.$$

A construction of convex liftings

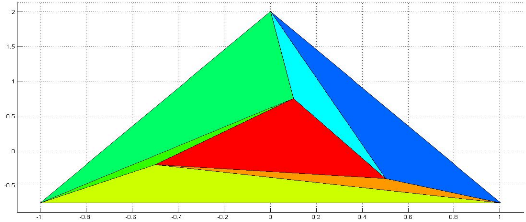
$$\boxed{\textbf{Feasibility} \longleftrightarrow \textbf{Convex liftability}}$$

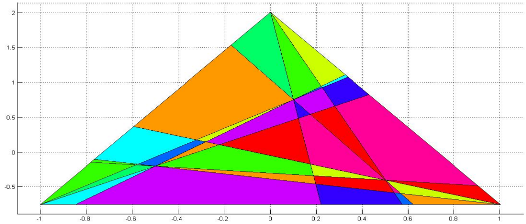
Nonconvexly liftable partitions

How to deal with non convexly liftable partitions?

Given a non convexly liftable polyhedral partition $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \subset \mathbb{R}^d$, there exists at least one subdivision, preserving the internal boundaries of this partition, such that the new cell complex is convexly liftable.

Nonconvexly liftable partitions





The **key point** is to use **hyperplane arrangement**

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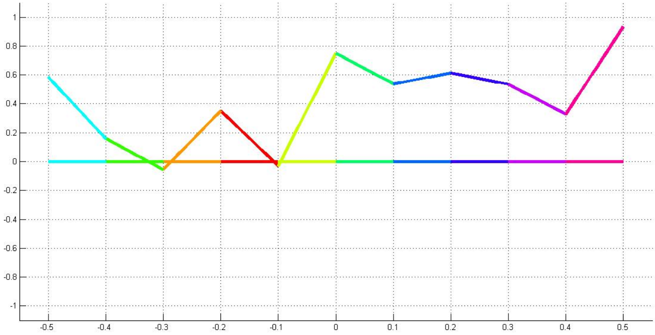
Solution to IPL/QP problem

Assumptions

- The given partition $\mathcal{X}$ is convexly liftable
- $\mathcal{X}$ is a partition of a polytope.

Constructive solution

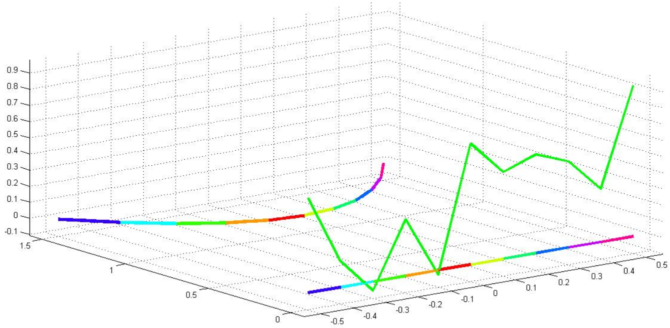
1. Construct a convex lifting $z(x)$ of the given cell complex $\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{X}_i$



Constructive solution

1. Construct a convex lifting $z(x)$ of the given cell complex $\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{X}_i$
2. Construct constraint set $\Pi_{\left[x^T\ z\ u^T\right]^T}$:

$$V_x = \bigcup_{i=1}^{N} \mathrm{vert}(\mathcal{X}_i), \;\; V_{\left[x^T\ z\right]^T} = \left\{ \begin{bmatrix} x \\ z(x) \end{bmatrix} \mid x \in V_x \right\}$$



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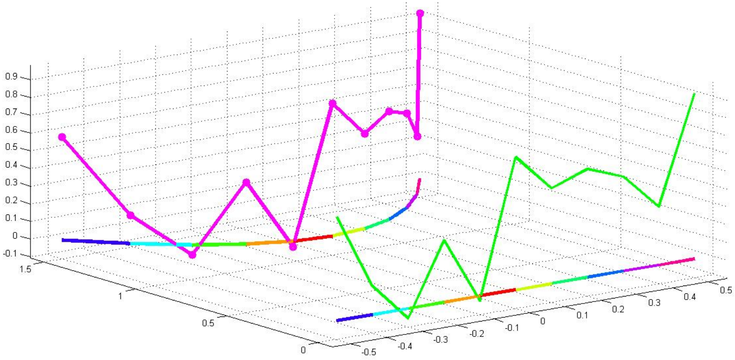
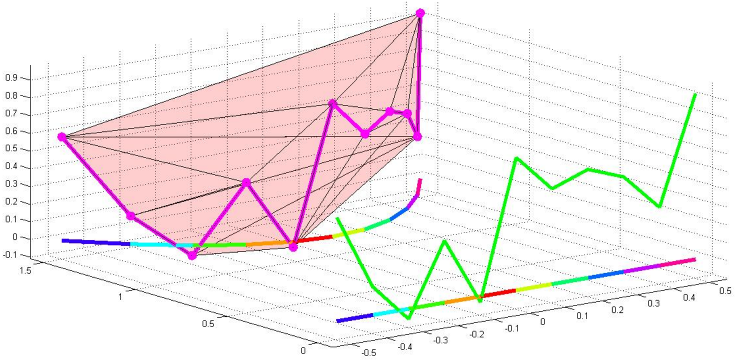
$$V_{\left[x^T\ z\ u^T\right]^T} = \left\{ \begin{bmatrix} x \\ z(x) \\ f_{pwa}(x) \end{bmatrix} \mid x \in V_x \right\}$$

$$\Pi_{\left[x^T\ z\ u^T\right]^T} = \mathrm{conv}\ V_{\left[x^T\ z\ u^T\right]^T}$$

Constructive solution



Constructive solution



Constructive solution

1. Construct a convex lifting $z(x)$ of the given polyhedral partition $\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{X}_i$.
2. Construct constraint set $\Pi_{\left[x^T \; z \; u^T\right]^T}$.
3. Compute the optimal solution to the optimization problem below:

$$\begin{bmatrix} z^* \\ u^* \end{bmatrix} = \arg \min_{\left[z \; u^T\right]^T} \quad z$$

$$\text{s.t.} \; \left[x^T \; z \; \; u^T\right]^T \in \Pi_{\left[x^T \; z \; \; u^T\right]^T}.$$

4. Obtain the original PWA function by restricting to the appropriate subcomponent of the above optimal vector:

$$u^* = \text{Proj}_{\mathbb{R}^{d_u}} \begin{bmatrix} z^* \\ u^* \end{bmatrix} = f_{pwa}(x).$$

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Related results

Parametric linear programming

The parameter space partition associated with an optimal solution to a parametric linear programming problem admits affinely equivalent polyhedra

Complexity of Parametric linear/quadratic programming

Any continuous PWA function defined over a polyhedral partition can be obtained by a parametric linear/quadratic programming problem with at most one supplementary 1-dimensional variable.

Linear MPC

Given a LTI system: $\boxed{x_{k+1} = Ax_k + Bu_k}$

Minimize:

$$\boxed{J(U, x_k) = \left[\begin{array}{l} \sum_{i=0}^{N-1} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} + x_{k+N|k}^T P x_{k+N|k} \\ \text{or} \\ \sum_{i=0}^{N-1} \|Qx_{k+i|k}\|_{1/\infty} + \|Ru_{k+i|k}\|_{1/\infty} + \|Px_{k+N|k}\|_{1/\infty} \end{array}\right.}$$

s.t. $$\boxed{\begin{array}{l} x_{k+i|k} \in \mathbb{X},\ u_{k+i|k} \in \mathbb{U} \text{ for } i = 0 \dots N-1 \\ x_{k+N|k} \in \mathbb{X}_T \end{array}}$$

Parametric L/Q programming

Minimize:

$$\boxed{J(\mathbf{u}, x_k) = \mathbf{u}^T H \mathbf{u} + (F x_k + C)^T \mathbf{u}.}$$

$$\text{s.t.} \qquad \boxed{G\mathbf{u} \leq E x_k + W}$$

$$\mathbf{u} = \begin{bmatrix} u_k^T & u_{k+1|k}^T & \cdots & u_{k+N-1|k}^T \end{bmatrix}^T$$

Solution

$$\begin{aligned} \mathbf{u}^*(x) = f_{pwa} : \bigcup_{i=1}^{N} \mathcal{X}_i \subset \mathbb{R}^{d_x} &\longrightarrow \mathbb{R}^{N \times d_u} \\ x \quad &\longmapsto f_i x + g_i \text{ for } x \in \mathcal{X}_i \end{aligned}$$

Applications to linear MPC

The continuous explicit solution of a generic linear MPC problem with respect to a linear/quadratic cost function is equivalently obtained through a linear MPC problem with a linear or quadratic cost function and the control horizon at most equal to 2 prediction steps.

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Double integrator 1/2

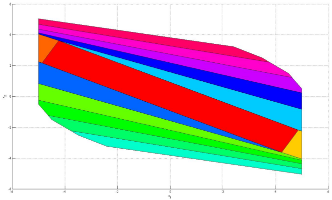
Double integrator model

$$x_{k+1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix} u_k \qquad Q = 10 * I_2, R = 0.5, N = 5$$

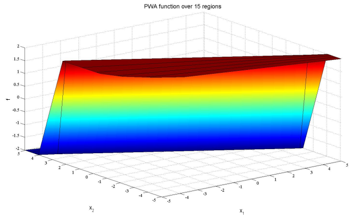
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \qquad -2 \leq u_k \leq 2, -5 \leq y_k \leq 5$$

$$J = x_{k+5|k}^T P x_{k+5|k} + \sum_{i=0}^{4} (x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k})$$

Polyhedral partition

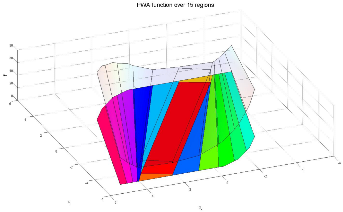


Associated PWA function

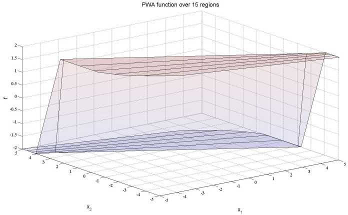


Double integrator 2/2

Result of convex lifting



Result of IOPCP

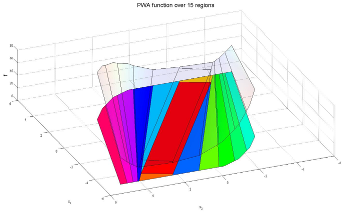


Equivalent formulations

Standard MPC problem	IOPCP
$$\min_{\mathbf{u}} J$$	$$\min_{\begin{bmatrix} z & u_k^T \end{bmatrix}^T} z$$
s.t: $-5 \leq \begin{bmatrix} 1 & 0 \end{bmatrix} x_{k+i\|k} \leq 5$ $-2 \leq u_{k+i\|k} \leq 2$ $0 \leq i \leq 4$ $x_{k+5\|k} \in \mathbb{X}_f$	s.t: $H \begin{bmatrix} x_k \\ z \\ u_k \end{bmatrix} \leq K,\ H \in \mathbb{R}^{24\times 4}, K \in \mathbb{R}^{24}$

Double integrator 2/2

Result of convex lifting



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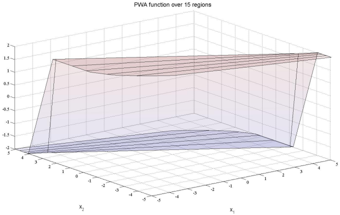


Equivalent formulations

Standard MPC problem

$$\min_{\mathbf{u}} J$$

$$\text{s.t:}\ -5 \leq \begin{bmatrix}1 & 0\end{bmatrix} x_{k+i|k} \leq 5$$
$$-2 \leq u_{k+i|k} \leq 2$$
$$0 \leq i \leq 4$$
$$x_{k+5|k} \in \mathbb{X}_f$$

IOPCP

$$\min_{\begin{bmatrix}u_{k+1} & u_k^T\end{bmatrix}^T} u_{k+1}$$

$$\text{s.t:}\quad H \begin{bmatrix} x_k \\ u_{k+1} \\ u_k \end{bmatrix} \leq K,\ H \in \mathbb{R}^{24\times 4},\ K \in \mathbb{R}^{24}$$

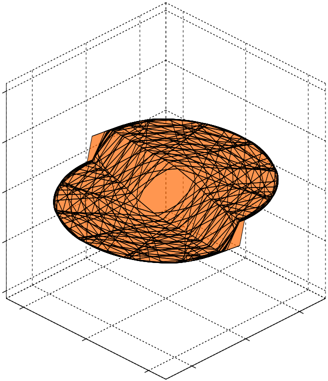
Case study - model of a lightly damped cantilever beam

M. Gulan, N. A. Nguyen, S. Olaru, P. Rodriguez-Ayerbe, B. Rohal-Ilkiv

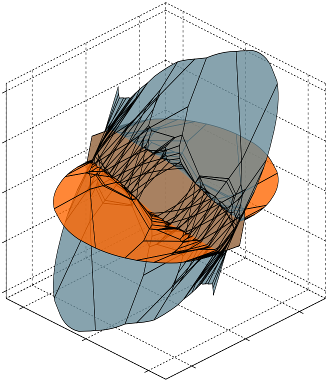
$$
\begin{aligned}
x_{k+1} &= \begin{bmatrix} 0.867 & 1.119 \\ -0.214 & 0.870 \end{bmatrix} x_k + \begin{bmatrix} 9.336\text{E-}4 \\ 5.309\text{E-}4 \end{bmatrix} u_k \\
y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k
\end{aligned}
\tag{3}
$$

with $|u| \leq 120$, $Q_x = C^T C$, and $Q_u = 1\text{E-}4$

Case-study example



(a) original PWA control law ($N = 40$, 3397 polyhedra)



(b) PWA obtained via IpLP vith clipping ($N = 2$, 811 polyhedra)

Case-study example

		task execution time [s]				
Formulation	$N =$	10	20	30	40	50
Standard MPC problem						
mpQP		6.3	24.2	91.6	171.9	356.2
Extended IpLP problem						
convex lifting		0.1	0.3	0.5	0.6	1.8
facet enumeration		0.05	0.07	0.12	0.17	0.23
constraint removal		2.8	14.9	38.3	74.3	135.3
mpLP		1.7	7.3	24.5	90.8	150.5

Thanks for your attention!