Navigation in a multi-obstacle environment. From partition of the space to a zonotopic-based MPC.

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GT - CPNL (Commande Prédictive Non Linéaire)

Outline

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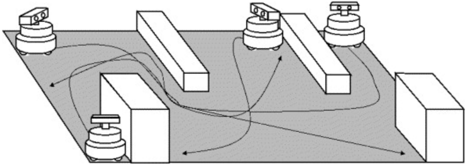
Motivation

Navigation through multi-obstacle environments is one of the most challenging and intensively studied problem in the control and robotics communities.

The main difficulty: the non-convexity of the feasible regions in the motion space and consequently the lack of connectivity in the solution space.

State-of-the-art methods:

- optimization-based (Chen et al. 2016; Janeček et al. 2017; Szmuk et al. 2017)
- sampled(graph)-based (LaValle 2006; Weiss et al. 2017)



Main ideas:

- consider efficient mixed-integer descriptions of the non-convex region(s)
- generate feasible paths based on space partitioning

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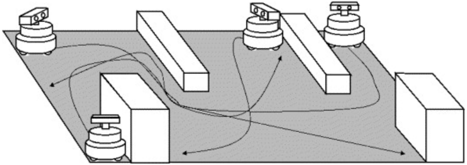

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Generic control strategy

Model Predictive Control (MPC) (Mayne et al. 2000; Maciejowski 2002)

$$J(x,u) = \left(k x_{k+N_p|k} \quad \bar{x}_{\mathrm{ref}\,|k} k^2_{\mathbf{P}} + \sum_{l=1}^{N_p-1} k x_{k+l|k} \quad \bar{x}_{\mathrm{ref}\,|k} k^2_{\mathbf{Q}} + \sum_{l=0}^{N_p-1} k \Delta u_{k+l|k} k^2_{\mathbf{R}} \right)$$

Ingredients:

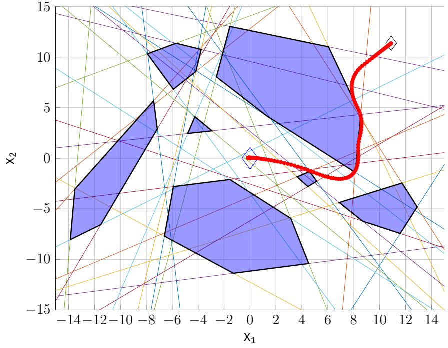
- quadratic/(non)linear optimization criterion
- (internal) model
- state and input constraints
    - magnitude constraints
    - obstacle avoidance constraints

$x_{k+l|k} = Ax_{k+l-1|k} + Bu_{k+l-1|k},$

$x_{k+l|k} \ 2 \ X, \ u_{k+l|k} \ 2 \ U,$

$x_{k+l|k} \ 2 \ P.$

- reference trajectory/path



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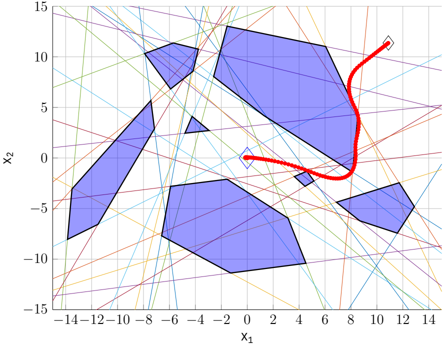

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Zonotopic representation

Idea: Use zonotopes to characterize the regions of interest (Althoff et al. 2010; Stoican et al. 2013)

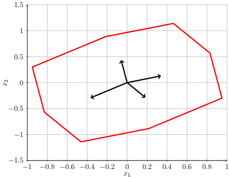
A zonotope is a centrally symmetric polytope and can be defined as a Minkowski sum of line segments (Kühn 1998)

$$Z = \{\mathbf{c} + \sum_{i=1}^{n_g} \xi_i \mathbf{g}_i : \|\xi\|_\infty \leq 1\}$$

$$Z = Z(G, c) = \{G\xi + c \mid \xi \in \mathbb{R}^{n_g}, \|\xi\|_\infty \leq 1\}$$

Several properties are of interest (Fukuda 2004).

- are closed under linear transformation and under Minkowski sum
- are symmetric,w.r.t. their centers
- their volume has an explicit formulation (Gover et al. 2010)
- detailed



They are increasingly used in control applications due to their numerical robustness and simplicity, well-suited for large-scale problems (Althoff 2015).

Zonotopic over-approximations I

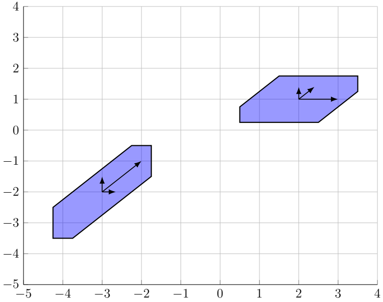
In order to efficiently over-approximate the given shape of the obstacles, we can parametrize the zonotopes (Althoff et al. 2010), with respect to some fixed direction (an a priori given matrix $G \in \mathbb{R}^{d\times m}$):

$$Z(G\Delta_j, c_j), \quad j = 1\dots N_o.$$

where Δ_ℓ is a diagonal matrix. The k-th diagonal element is noted as δ_{j_k}.

Consequences:

- the generated half-spaces share common normal vectors irrespective of the scaling factors.
- the resulting hyperplanes are parallel with each other
- simplified formulation



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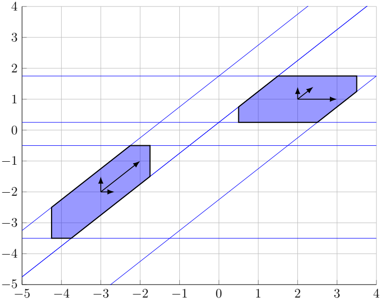
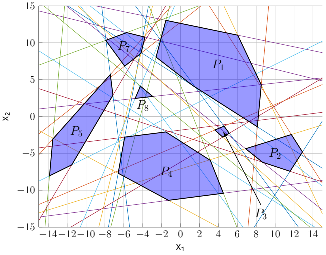


Zonotopic over-approximations II

Goal: Provide adequate zonotopic over-approximations for a multi-obstacle environment:



#P	#H	$\gamma^*(N)$	$t_{\gamma(N)}$(sec)	$\#\Sigma_P$
7	34	419	9.22	75

$$(\Delta_j, c_j)^* = \arg\min_{\Delta_j, c_j} C(\Delta_j, c_j)$$

$$\text{s.t. } P_j \quad Z(G\Delta_j, c_j)$$

Measures $C(\Delta_j, c_j)$:

i) $\mathrm{Vol}(Z(G\Delta_j, c_j))$ -zonotope volume

ii) $\|\delta\|_1$ -generator sum $(\sum_{k=1}^{m} g_k \delta_{j_k})$

iii) $\|\delta\|_\infty$ -largest generator $(\max_{k=1\ldots m} g_k \delta_{j_k})$

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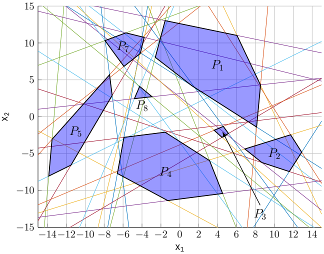

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Measures $C(\Delta_j, c_j)$:

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Illustrative example for zonotopic representation I

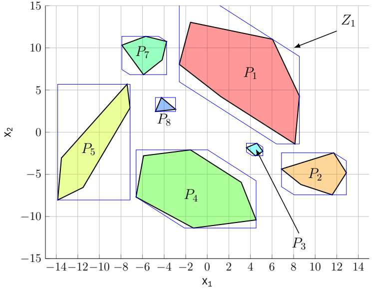


Figure: Volume with $G = G_3$.

$$G_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

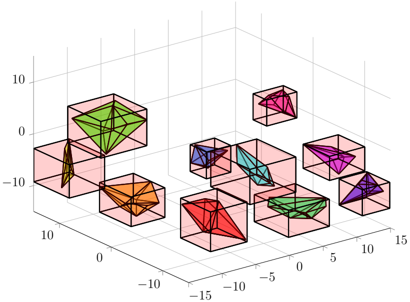


Figure: Volume with $G = G_4$.

$$G_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Illustrative example for zonotopic representation II

	Measure	G	t_{sol}	#H	$\gamma^*(N)$	$\frac{\Delta\gamma(N)}{\gamma(N)}$	$t_{\gamma^*(N)}$(sec)	$\#\Sigma_P$	V	$\frac{\Delta V}{V}$(%)
$d=2$	$k\delta k_1$	G_1	8,13	42	505	20,53	9,53	197	**376,98**	71,7
		G_2	8,02	28	225	-46,30	3,81	101	410,07	86,78
		G_3	8,27	42	534	27,45	10,09	167	368,93	68,04
	$k\delta k_\infty$	G_1	8,19	42	441	5,25	8,19	374	897,92	308,98
		G_2	8,01	28	225	-46,30	3,91	175	583,33	165,69
		G_3	8,19	42	441	5,25	8,19	374	897,92	308,98
	Vol	G_1	9,40	42	510	21,72	9,66	199	**368,41**	67,8
		G_2	9,19	28	225	-46,30	3,75	101	410,07	86,78
		G_3	9,30	40	530	26,49	10,27	169	374,99	70,8
$d=3$	$k\delta k_1$	G_4	9,82	60	8400	-85,31	105,98	934	**1857,46**	323,56
		G_5	10,50	120	62480	9,26	1145,09	4952	2019,34	360,47
	$k\delta k_\infty$	G_4	9,71	60	8000	-86,01	100,22	1127	2623,3	498,19
		G_5	10,59	120	51396	-10,12	932,12	9432	5852,55	1234,56
	Vol	G_4	11,02	60	8400	-85,31	105,07	934	**1857,46**	323,56
		G_5	11,79	84	24528	-57,11	413,07	2218	1908,2	335,13

The fixed directions/Generators:

$$G_{1,2,3}\ 2 \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right\},\ G_{4,5}\ 2 \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\}.$$

Obstacle avoidance: polytopic vs. zonotopic representation

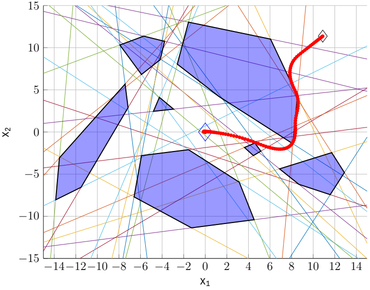


Figure: Polytopes P.

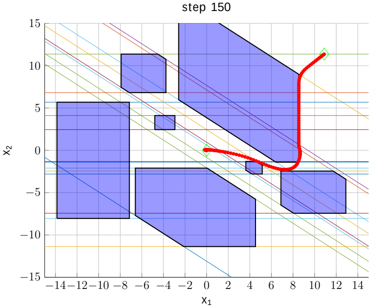
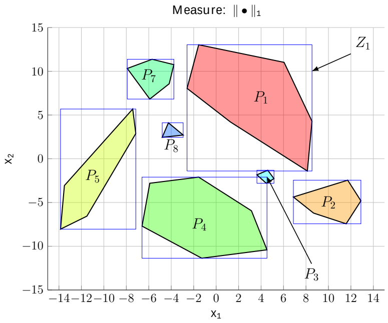


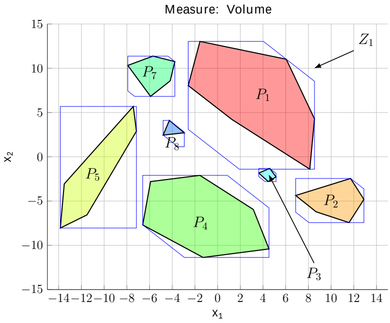
Figure: Zonotopes Z.

	Topology	N_{goal}	t_{goal}(sec)	t_{worst} (sec)
$d = 2$	P	143	11.64	0.22
	Z	146	10.07	0.18
$d = 3$	P	98	83.87	0.81
	Z	132	57.07	0.42

Table: MPC parameters: $N_p = 10$, $P = 10I_{2d}$, $Q = I_{2d}$, $R = I_d$.



$G = G_2$



$G = G_1$

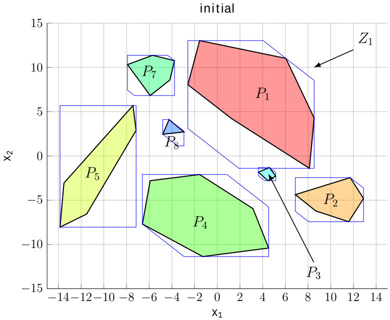
"How can we approximate the obstacles with zonotopic sets while, simultaneously, safeguarding the feasible paths of the initial problem?"

Zonotopic approximations with corridors

Proposed solution: The inclusion of a separation hyperplane (Boyd et al. 2004) in the optimization problem: $H_{sep} = \{x \in \mathbb{R}^d : h_{sep}^\top x = k_{sep}\}$, with its corresponding half-spaces R_{sep}^{+} and R_{sep}^{-}.

- adding a linear constraint: $Z(G\Delta, c) \quad R_{sep}^{\pm}$
- Shortcoming: usually, infeasible

- adding generators spanning H_{sep}
- Shortcoming: no inclusion monotonicity



Theorem

In $\mathbb{R}^d$, the maximum number of joint constraints for corridors with feasibility guarantees is $d+1$.

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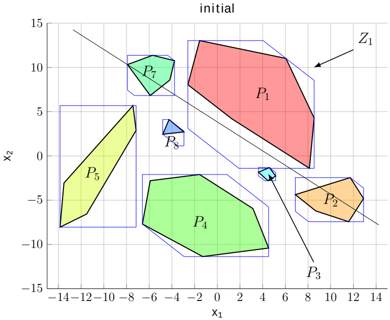

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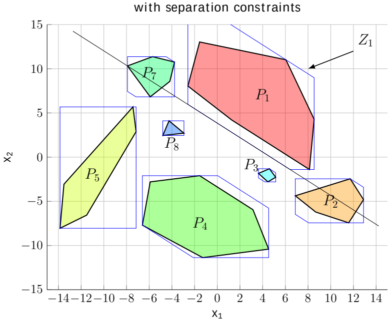

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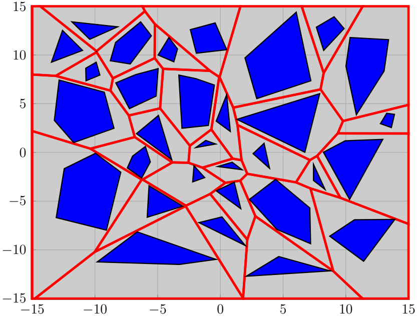
Space Partitioning

Idea: Provide a partitioning of the navigation space w.r.t. the obstacles $\mathsf{P} = \bigcup_{i=1}^{N_o} P_i$



Definition

A family of sets $\{X_i\}_{i\in\mathcal{I}}$ verifying:

i) $\mathsf{X} = \bigcup_{i=1}^{N_o} X_i$,

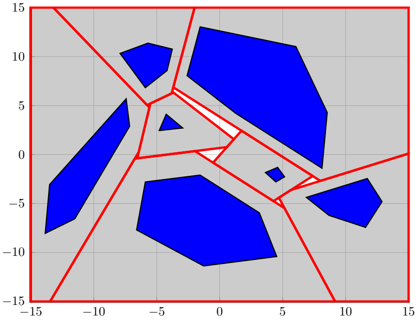
ii) $\mathrm{int}(X_i)\bigcap \mathrm{int}(X_j) = \emptyset, \forall i \neq j \in \mathcal{I}$,

iii) $P_i \subseteq \mathrm{int}(X_i), \forall i \in \{1\dots, N_o\}$

is called a partition of X induced by the obstacles P.

An intuitive solution

Idea: As $P_i \cap P_j = \emptyset, \forall i \neq j$, the separating hyperplanes(Boyd et al. 2004) are candidates for the supporting hyperplanes of X_i P_i



$X \neq \bigcup_{i=1}^{N_o} X_i$

Alternatives:

- Generalized Voronoi Diagram (Afonso et al. 2013) or (Sugihara 1993)
- grid of square/cubic cells (Wang et al. 2015)
- convex lifting (Nguyen et al. 2018)

[1] Nguyen, N. A., Gulan, M., Olaru, S., Rodriguez-Ayerbe, P. (2018). Convex lifting: Theory and control applications. IEEE Transactions on Automatic Control, 63(5), 1243-1258.

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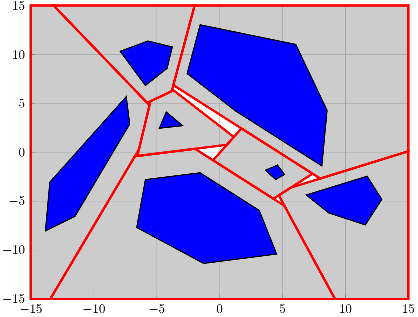

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Convex lifting-overview

Definition (Convex lifting Nguyen et al. 2018)

For a polyhedral partition $fX_i g_{i \in \mathcal{I}}$ of a domain X, a piecewise affine lifting described by the function:

$$z(x) = a_i^\top x + b_i, x \ 2 \ X_i,$$

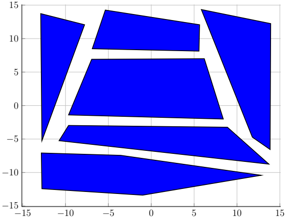
is called a convex (piecewise affine) lifting, if $z(x)$ is continuous and convex over partition $fX_i g_{i \in \mathcal{I}}$ of X.

$$\min_{a_i, b_i} \sum_{i=1}^{N_o} J(a_i, b_i) = \sum_{i=1}^{N_o} \begin{bmatrix} a_i^\top & b_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$

$$\text{s.t.} \quad a_i^\top v + b_i \quad M, 8v \ 2 \ V(X_i), 8i,$$

$$a_j^\top \nu + b_j \quad a_i^\top \nu + b_i + \epsilon, 8\nu \ 2 \ V(X_j) \, n \, V(X_i), 8i \ 6= j,$$

$$a_i^\top v + b_i = a_j^\top v + b_j, 8v \ 2 \ V(X_i \ \backslash \ X_j), 8i \ 6= j.$$



$$P = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \ 2 \ \mathbb{R}^{d+1} : \begin{bmatrix} a_i^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad b_i, \ i \ 2 \ I \right\}.$$

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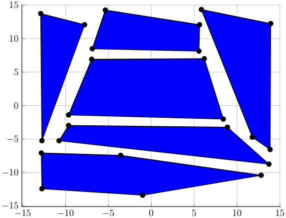

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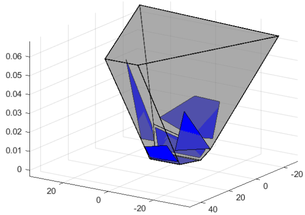

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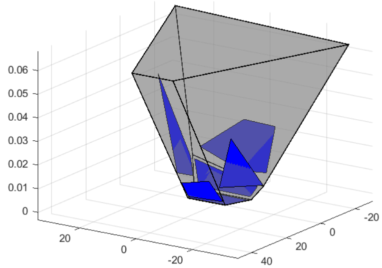
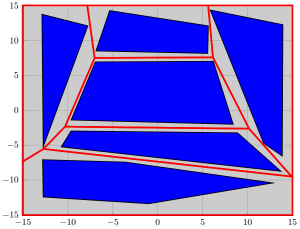
From Lifting to Partitioning

projections of the facets on X

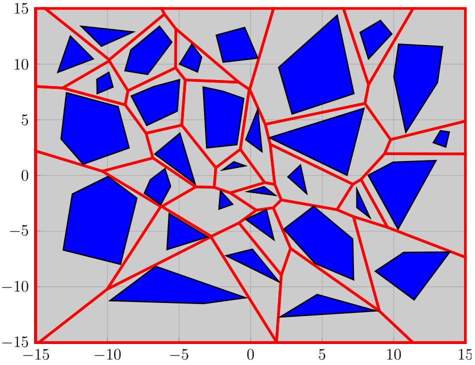
$$P = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} 2 \; \mathrm{R}^{d+1} : \begin{bmatrix} a_i^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad b_i, \; i \; 2 \; I \right\}.$$

$$f X_i g_{i \in \mathcal{I}}$$

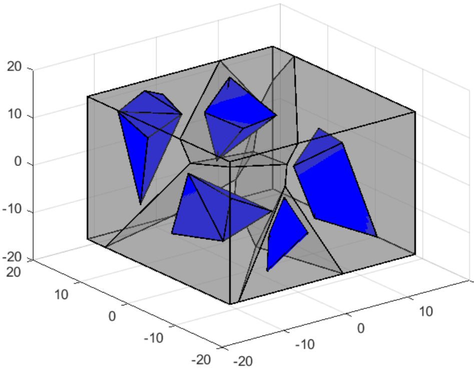




Space partitioning



$N_o = 31,\ d = 2$

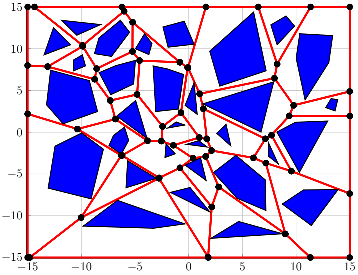


$N_o = 5,\ d = 3$

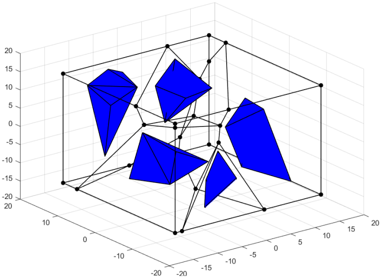
Geometric path generation

Idea: Define a graph $\Gamma = (N, E, f)$, $f : E \to \mathbb{R}$, based on the partition $\{X_i\}_{i=1:N_o}$.

- $N = \{V(X_i)\}_{i=1:N_o}$
- E - the facets of the partition cells.



$N_o = 31,\ d = 2$



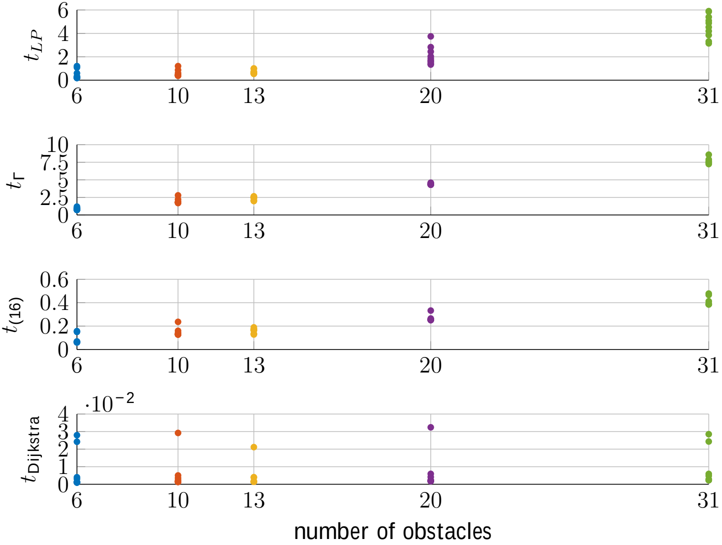
$N_o = 5,\ d = 3$

Next steps:

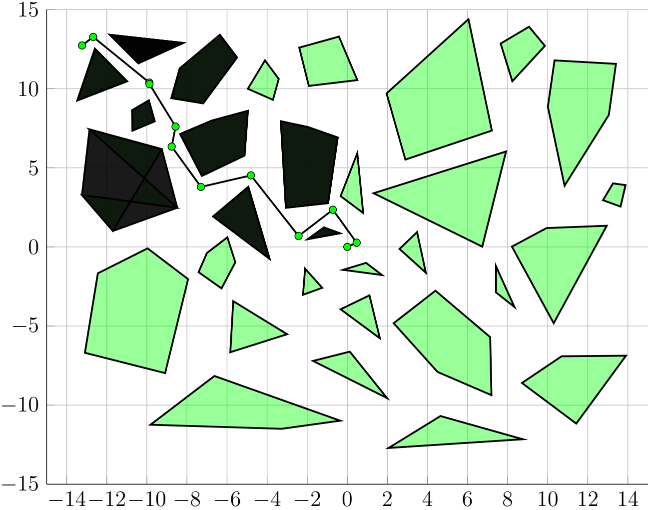
- connect $x_0, x_f \in \mathcal{C}_X(P)$ to Γ
- run a graph search algorithm, e.g. Dijkstra's Algorithm (Karaman et al. 2011)

Monte Carlo study for feasible path generation

Computing times w.r.t. the number of obstacles



Illustrative example for obstacle avoidance



- $fX_ig_{i=1:N_o}$ and Γ
- “local” zonotopic over-approximation
- Path(x_i, x_f)
- MPC with $N_p = 7$

Outline

Conclusions

Conclusions:

- we studied the benefits and difficulties of choosing a particular family of sets (parametrized zonotopes) for the non-convex feasible domain representation
- we introduced a partitioning procedure of the workspace based on convex lifting
- we propose a navigation strategy with obstacle avoidance guarantees using local zonotopic approximations of the obstacles

Future directions:

- extension of the preliminary results for more complex scenarios (e.g., mobile obstacles)
- improvements of the MPC problem (feasibility etc.)
- develop a *complete* navigation strategy based on space partitioning

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Zonotopic sets { properties

Several properties are of interest (Fukuda 2004).

Let $Z_1 = Z(\mathbf{G}_1, \mathbf{c}_1) \quad \mathbb{R}^n$, $Z_1 = Z(\mathbf{G}_2, \mathbf{c}_2) \quad \mathbb{R}^n$:

- are closed under linear transformation: $\mathbf{R}Z(\mathbf{G}_1, \mathbf{c}_1) = Z(\mathbf{R}\mathbf{G}_1, \mathbf{R}\mathbf{c}_1)$;
- are closed under Minkowski sum: $Z(\mathbf{G}_1, \mathbf{c}_1) \quad Z(\mathbf{G}_2, \mathbf{c}_2) = Z(\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}, \mathbf{c}_1 + \mathbf{c}_2)$;
- are symmetric, w.r.t. their center: $Z_1 = \quad Z f \mathbf{G}_1, \mathbf{c}_1 g = Z f \mathbf{G}_1, \quad \mathbf{c}_1 g$
- their volume has an explicit formulation (Gover et al. 2010): $\mathrm{Vol}(Z(G,c)) = \sum_{1 \le j_1 < j_2 \dots j_d \le m} \left|\det(G^{j_1 \cdots j_d})\right|$
- their corresponding half-space representation (Althoff et al. 2010):

$$Z(G,c) = \bigcap_{1 \le j_1 < \dots j_{d-1} \le m} f x \ 2 \ \mathbb{R}^d : j h_i(x \quad c) j \quad k_i g,$$

$$h_i \ ? \ g_{j_l}, 8 j_l \ 2 \ f j_1 \dots j_{d-1} g, \quad k_i = \sum_{j_l \notin \{j_1 \cdots j_{d-1}\}} \left| h_i^\top g_{j_l} \right|$$

Back to Zonotopic Approximation.