Navigation in a multi-obstacle environment. From partition of the space to a zonotopic-based MPC.

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GT - CPNL (Commande Prédictive Non Linéaire)

Outline

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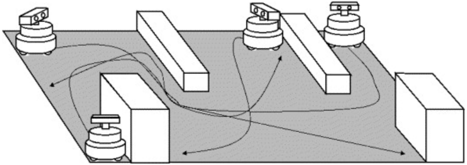
Motivation

Navigation through multi-obstacle environments is one of the most challenging and intensively studied problem in the control and robotics communities.

The main difficulty: the non-convexity of the feasible regions in the motion space and consequently the lack of connectivity in the solution space.

State-of-the-art methods:

- optimization-based (Chen et al. 2016; Janeček et al. 2017; Szmuk et al. 2017)
- sampled(graph)-based (LaValle 2006; Weiss et al. 2017)



Main ideas:

- consider efficient mixed-integer descriptions of the non-convex region(s)
- generate feasible paths based on space partitioning

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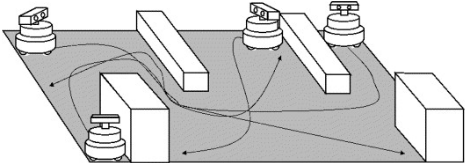

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Generic control strategy

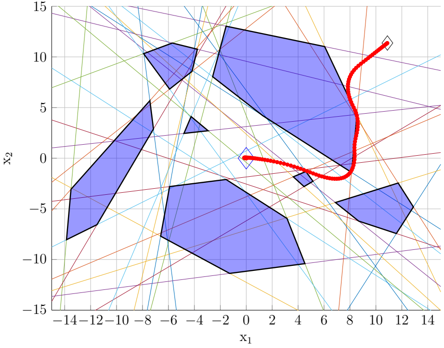
Model Predictive Control (MPC) (Mayne et al. 2000; Maciejowski 2002)

$$\mathcal{J}(x,u) = \left(\|x_{k+N_p|k} - \bar{x}_{\mathrm{ref}|k}\|^2_{\mathbf{P}} + \sum_{l=1}^{N_p-1} \|x_{k+l|k} - \bar{x}_{\mathrm{ref}|k}\|^2_{\mathbf{Q}} + \sum_{l=0}^{N_p-1} \|\Delta u_{k+l|k}\|^2_{\mathbf{R}} \right)$$

Ingredients:

- quadratic/(non)linear optimization criterion
- (internal) model
- state and input constraints
  - magnitude constraints
  - obstacle avoidance constraints

$$x_{k+l|k} = Ax_{k+l-1|k} + Bu_{k+l-1|k},$$
$$x_{k+l|k} \in \mathcal{X}, u_{k+l|k} \in \mathcal{U},$$
$$x_{k+l|k} \notin \mathbb{P}.$$



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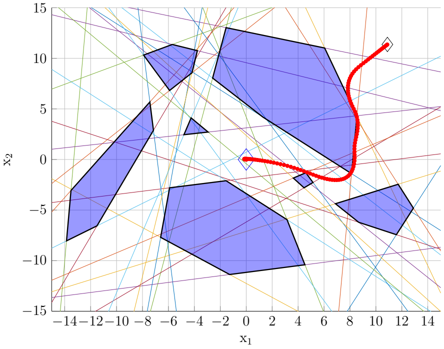

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Zonotopic representation

Idea: Use zonotopes to characterize the regions of interest (Althoff et al. 2010; Stoican et al. 2013)

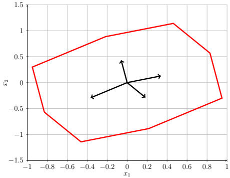
A zonotope is a centrally symmetric polytope and can be defined as a Minkowski sum of line segments (Kühn 1998)

$$Z = \{\mathbf{c} + \sum_{i=1}^{n_g} \xi_i \mathbf{g}_i : \|\xi\|_\infty \leq 1\}$$

$$Z = \mathcal{Z}(G, c) = \{G\xi + c | \xi \in \mathbb{R}^{n_g}, \|\xi\|_\infty \leq 1\}$$

Several properties are of interest (Fukuda 2004).

- are closed under linear transformation and under Minkowski sum
- are symmetric,w.r.t. their centers
- their volume has an explicit formulation (Gover et al. 2010)
- detailed



They are increasingly used in control applications due to their numerical robustness and simplicity, well-suited for large-scale problems (Althoff 2015).

Zonotopic over-approximations I

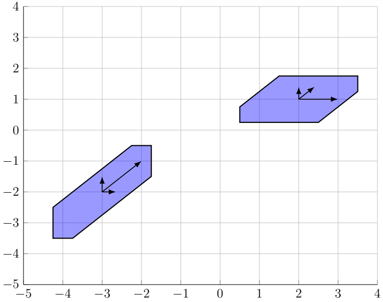
In order to efficiently over-approximate the given shape of the obstacles, we can parametrize the zonotopes (Althoff et al. 2010), with respect to some fixed direction (an a priori given matrix $G \in \mathbb{R}^{d \times m}$):

$$\mathcal{Z}(G\Delta_j, c_j), \quad j = 1 \dots N_o.$$

where Δ_ℓ is a diagonal matrix. The k-th diagonal element is noted as δ_{j_k}.

Consequences:

- the generated half-spaces share common normal vectors irrespective of the scaling factors.
- the resulting hyperplanes are parallel with each other
- simplified formulation



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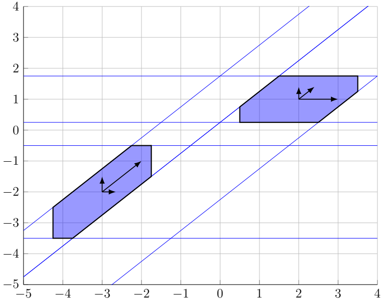
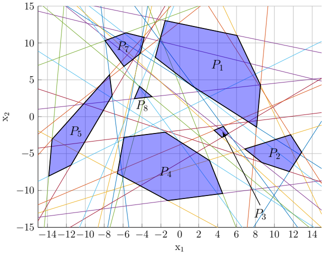


Zonotopic over-approximations II

Goal: Provide adequate zonotopic over-approximations for a multi-obstacle environment:



$\#\mathbb{P}$	$\#\mathbb{H}$	$\gamma^*(N)$	$t_{\gamma(N)}$(sec)	$\#\Sigma_{\mathbb{P}}$
7	34	419	9.22	75

$$(\Delta_j, c_j)^* = \arg\min_{\Delta_j, c_j} \mathcal{C}(\Delta_j, c_j)$$
$$\text{s.t. } P_j \subseteq \mathcal{Z}(G\Delta_j, c_j)$$

Measures $\mathcal{C}(\Delta_j, c_j)$:

i) $\text{Vol}(\mathcal{Z}(G\Delta_j, c_j))$ -zonotope volume

ii) $\|\delta\|_1$ -generator sum $(\sum_{k=1}^{m} g_k \delta_{j_k})$

iii) $\|\delta\|_\infty$ -largest generator $(\max_{k=1\ldots m} g_k \delta_{j_k})$

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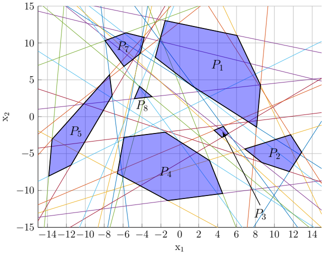

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$$\text{s.t.} \quad \left| h_i^\top (v_k - c_j) \right| \leq k_i(\Delta_j), \ \forall k.$$

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Illustrative example for zonotopic representation I

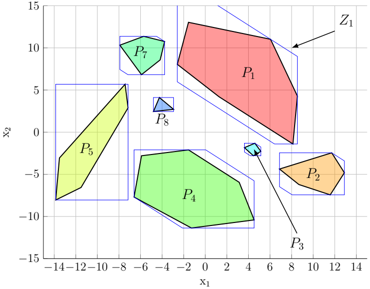


Figure: Volume with $G = G_3$.

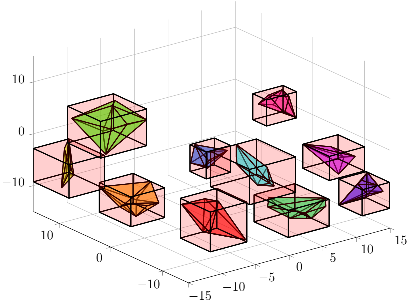


Figure: Volume with $G = G_4$.

$$G_3 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Illustrative example for zonotopic representation II

	Measure	G	t_{sol}	$\#\mathbb{H}$	$\gamma^*(N)$	$\frac{\Delta\gamma(N)}{\gamma(N)}$	$t_{\gamma^*(N)}$(sec)	$\#\Sigma_{\mathbb{P}}$	V	$\frac{\Delta V}{V}$(%)
$d=2$	$\|\delta\|_1$	G_1	8,13	42	505	20,53	9,53	197	**376,98**	71,7
		G_2	8,02	28	225	-46,30	3,81	101	410,07	86,78
		G_3	8,27	42	534	27,45	10,09	167	368,93	68,04
	$\|\delta\|_\infty$	G_1	8,19	42	441	5,25	8,19	374	897,92	308,98
		G_2	8,01	28	225	-46,30	3,91	175	583,33	165,69
		G_3	8,19	42	441	5,25	8,19	374	897,92	308,98
	Vol	G_1	9,40	42	510	21,72	9,66	199	**368,41**	67,8
		G_2	9,19	28	225	-46,30	3,75	101	410,07	86,78
		G_3	9,30	40	530	26,49	10,27	169	374,99	70,8
$d=3$	$\|\delta\|_1$	G_4	9,82	60	8400	-85,31	105,98	934	**1857,46**	323,56
		G_5	10,50	120	62480	9,26	1145,09	4952	2019,34	360,47
	$\|\delta\|_\infty$	G_4	9,71	60	8000	-86,01	100,22	1127	2623,3	498,19
		G_5	10,59	120	51396	-10,12	932,12	9432	5852,55	1234,56
	Vol	G_4	11,02	60	8400	-85,31	105,07	934	**1857,46**	323,56
		G_5	11,79	84	24528	-57,11	413,07	2218	1908,2	335,13

The fixed directions/Generators:

$$G_{1,2,3} \in \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \right\},\ G_{4,5} \in \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\}.$$

Obstacle avoidance: polytopic vs. zonotopic representation

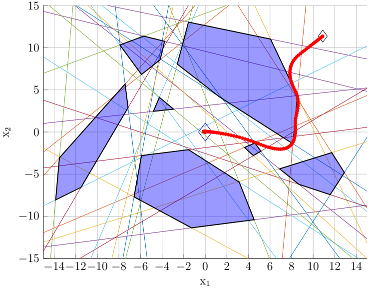


Figure: Polytopes $\mathbb{P}$.

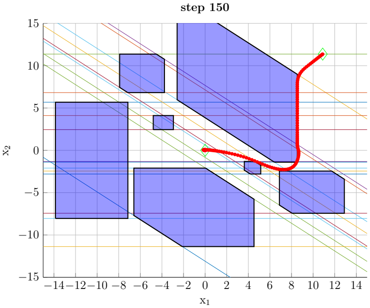
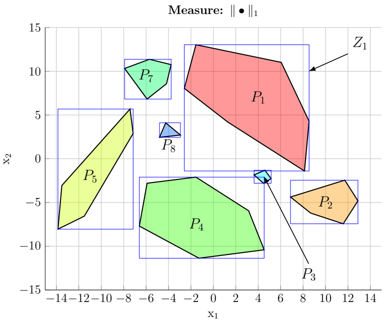


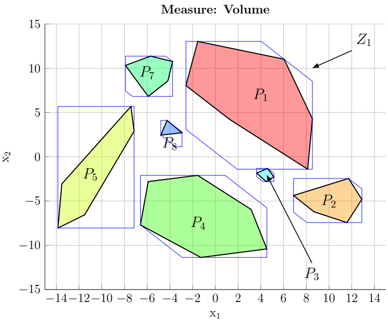
Figure: Zonotopes $\mathbb{Z}$.

	Topology	N_{goal}	t_{goal}(sec)	t_{worst} (sec)
$d = 2$	$\mathbb{P}$	143	11.64	0.22
	$\mathbb{Z}$	146	10.07	0.18
$d = 3$	$\mathbb{P}$	98	83.87	0.81
	$\mathbb{Z}$	132	57.07	0.42

Table: MPC parameters: $N_p = 10$, $P = 10I_{2d}$, $Q = I_{2d}$, $R = I_d$.



$G = G_2$



$G = G_1$

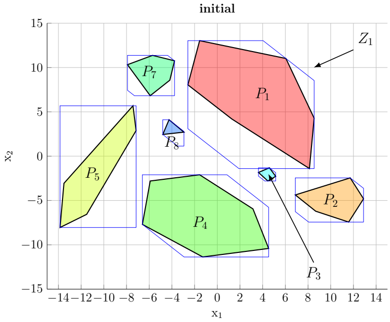
"How can we approximate the obstacles with zonotopic sets while, simultaneously, safeguarding the feasible paths of the initial problem?"

Zonotopic approximations with corridors

Proposed solution: The inclusion of a separation hyperplane (Boyd et al. 2004) in the optimization problem: $\mathcal{H}_{sep} = \{x \in \mathbb{R}^d : h_{sep}^\top x = k_{sep}\}$, with its corresponding half-spaces $\mathcal{R}_{sep}^+$ and $\mathcal{R}_{sep}^-$.

- adding a linear constraint: $\mathcal{Z}(G\Delta, c) \subset \mathcal{R}_{sep}^{\pm}$
- Shortcoming: usually, infeasible

- adding generators spanning $\mathcal{H}_{sep}$
- Shortcoming: no inclusion monotonicity



Theorem

In $\mathbb{R}^d$, the maximum number of joint constraints for corridors with feasibility guarantees is $d + 1$.

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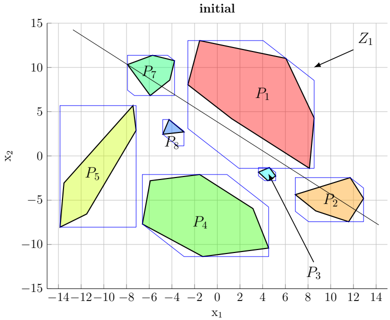

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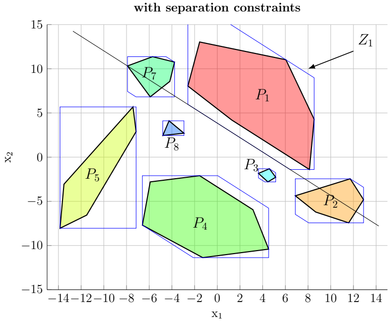

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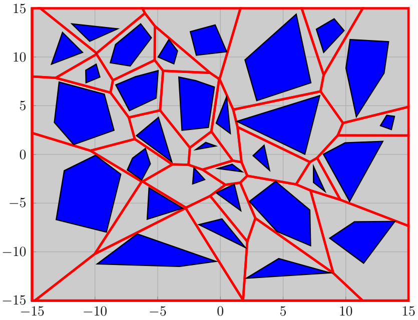
Space Partitioning

Idea: Provide a partitioning of the navigation space w.r.t. the obstacles $\mathbb{P} = \bigcup_{i=1}^{N_o} P_i$



Definition

A family of sets $\{X_i\}_{i\in\mathcal{I}}$ verifying:

i) $\mathbb{X} = \bigcup_{i=1}^{N_o} X_i$,

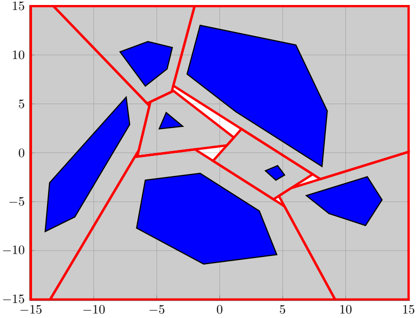
ii) $\text{int}(X_i) \bigcap \text{int}(X_j) = \emptyset, \forall i \neq j \in \mathcal{I}$,

iii) $P_i \subset \text{int}(X_i), \forall i \in \{1 \ldots, N_o\}$

is called a partition of $\mathbb{X}$ induced by the obstacles $\mathbb{P}$.

An intuitive solution

Idea: As $P_i \cap P_j = \emptyset, \forall i \neq j$, the separating hyperplanes(Boyd et al. 2004) are candidates for the supporting hyperplanes of $X_i \supset P_i$



$\mathbb{X} \neq \bigcup_{i=1}^{N_o} X_i$

Alternatives:

- Generalized Voronoi Diagram (Afonso et al. 2013) or (Sugihara 1993)
- grid of square/cubic cells (Wang et al. 2015)
- convex lifting (Nguyen et al. 2018)

[1] Nguyen, N. A., Gulan, M., Olaru, S., Rodriguez-Ayerbe, P. (2018). Convex lifting: Theory and control applications. IEEE Transactions on Automatic Control, 63(5), 1243-1258.

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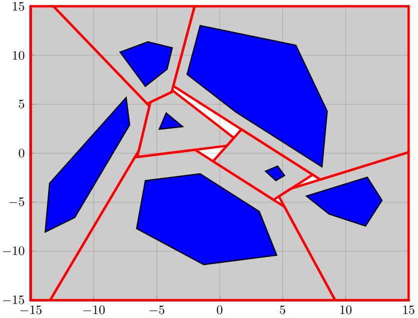

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Convex lifting-overview

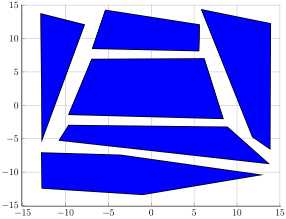
Definition (Convex lifting Nguyen et al. 2018)

For a polyhedral partition $\{X_i\}_{i\in\mathcal{I}}$ of a domain $\mathcal{X}$, a piecewise affine lifting described by the function:

$$z(x) = a_i^\top x + b_i, x \in X_i,$$

is called a convex (piecewise affine) lifting, if $z(x)$ is continuous and convex over partition $\{X_i\}_{i\in\mathcal{I}}$ of $\mathcal{X}$. ♦

$$\min_{a_i, b_i} \sum_{i=1}^{N_o} J(a_i, b_i) = \sum_{i=1}^{N_o} \begin{bmatrix} a_i^\top & b_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$
$$\text{s.t.} \quad a_i^\top v + b_i \leq M, \forall v \in \mathcal{V}(X_i), \forall i,$$
$$a_j^\top \nu + b_j \geq a_i^\top \nu + b_i + \epsilon, \forall \nu \in \mathcal{V}(X_j) \setminus \mathcal{V}(X_i), \forall i \neq j,$$
$$a_i^\top v + b_i = a_j^\top v + b_j, \forall v \in \mathcal{V}(X_i \cap X_j), \forall i \neq j.$$



$$\mathcal{P} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} : \begin{bmatrix} a_i^\top & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq -b_i, \; i \in \mathcal{I} \right\}.$$

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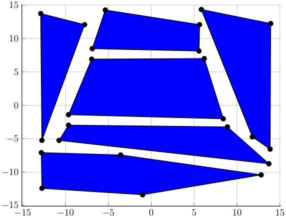

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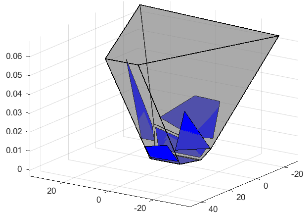
$$\text{s.t.} \quad a_i^\top v + b_i \leq M, \forall v \in \mathcal{V}(P_i), \forall i,$$
$$a_i^\top v + b_i \geq a_j^\top v + b_j + \epsilon, \forall v \in \mathcal{V}(P_i), \forall i \neq j,$$



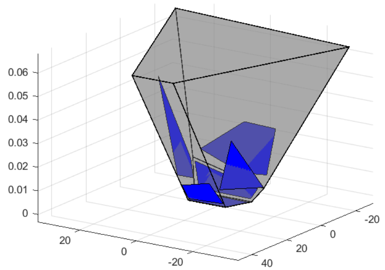
$$\mathcal{P} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} : \begin{bmatrix} a_i^\top & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq -b_i, \ i \in \mathcal{I} \right\}.$$

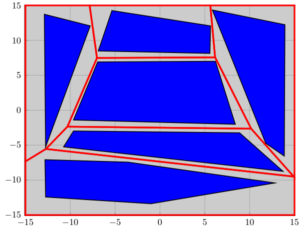
From Lifting to Partitioning

projections of the facets on $\mathbb{X}$

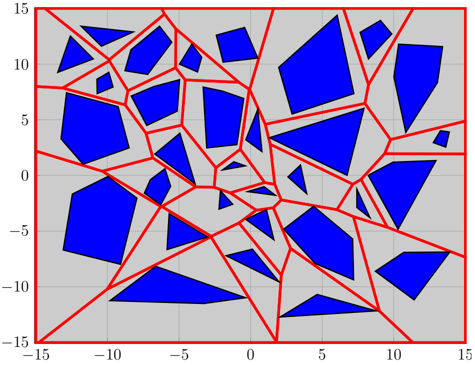
$$\mathcal{P} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{d+1} : \begin{bmatrix} a_i^\top & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \leq -b_i,\ i \in \mathcal{I} \right\}.$$

$\{X_i\}_{i\in\mathcal{I}}$

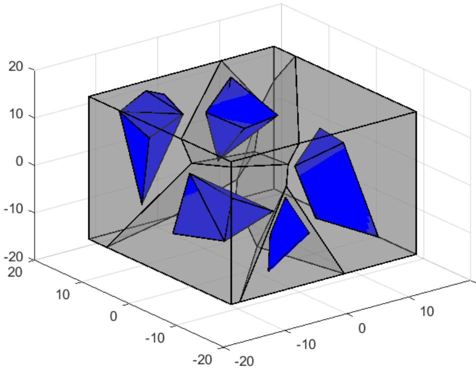




Space partitioning



$N_o = 31, \ d = 2$

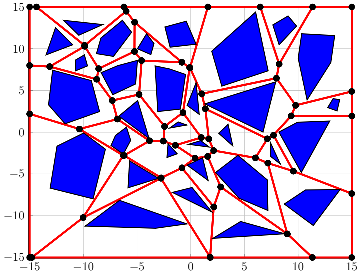


$N_o = 5, \ d = 3$

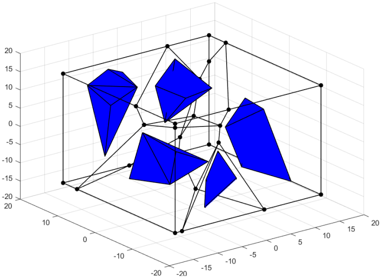
Geometric path generation

Idea: Define a graph $\Gamma = (\mathcal{N}, \mathcal{E}, f)$, $f : \mathcal{E} \to \mathbb{R}$, based on the partition $\{X_i\}_{i=1:N_o}$.

- $\mathcal{N} = \{\mathcal{V}(X_i)\}_{i=1:N_o}$
- $\mathcal{E}$ - the facets of the partition cells.



$N_o = 31,\ d = 2$



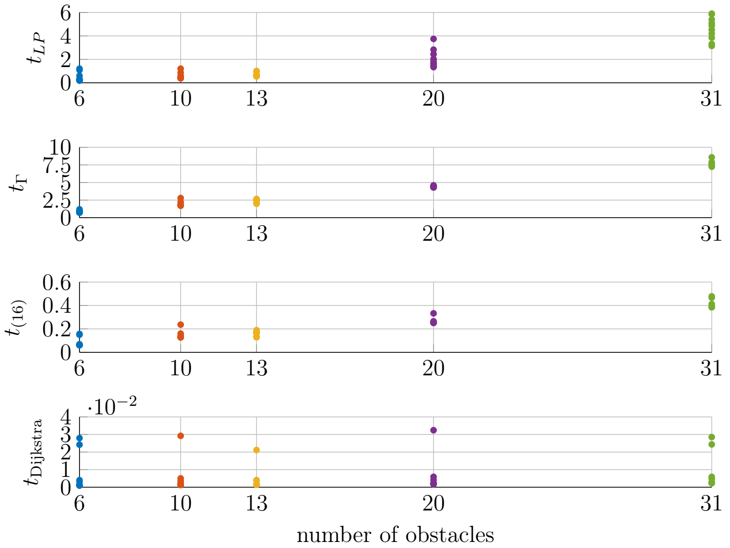
$N_o = 5,\ d = 3$

Next steps:

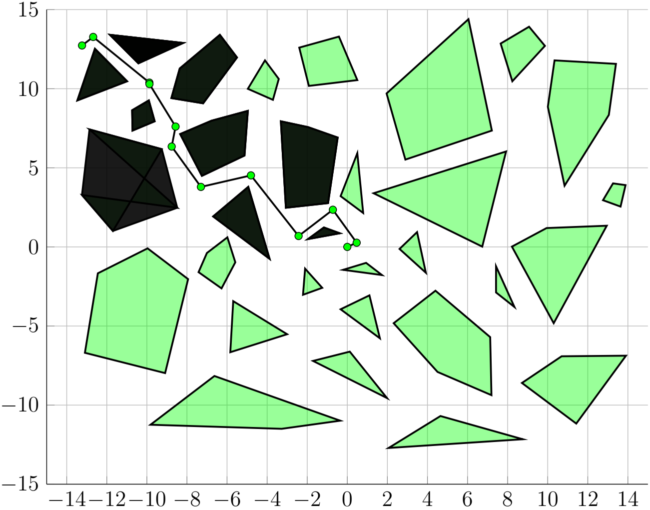
- connect $x_0, x_f \in \mathcal{C}_{\mathbb{X}}(\mathbb{P})$ to Γ
- run a graph search algorithm, e.g. Dijkstra's Algorithm (Karaman et al. 2011)

Monte Carlo study for feasible path generation

Computing times w.r.t. the number of obstacles



Illustrative example for obstacle avoidance



- $\{X_i\}_{i=1:N_o}$ and Γ
- "local" zonotopic over-approximation
- $\mathrm{Path}(x_i, x_f)$
- MPC with $N_p = 7$

Outline

Conclusions

Conclusions:

- we studied the benefits and difficulties of choosing a particular family of sets (parametrized zonotopes) for the non-convex feasible domain representation
- we introduced a partitioning procedure of the workspace based on convex lifting
- we propose a navigation strategy with obstacle avoidance guarantees using local zonotopic approximations of the obstacles

Future directions:

- extension of the preliminary results for more complex scenarios (e.g., mobile obstacles)
- improvements of the MPC problem (feasibility etc.)
- develop a *complete* navigation strategy based on space partitioning

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Zonotopic sets – properties

Several properties are of interest (Fukuda 2004).

Let $Z_1 = \mathcal{Z}(\mathbf{G}_1, \mathbf{c}_1) \subset \mathbb{R}^n$, $Z_1 = \mathcal{Z}(\mathbf{G}_2, \mathbf{c}_2) \subset \mathbb{R}^n$:

- are closed under linear transformation: $\mathbf{R}\mathcal{Z}(\mathbf{G}_1, \mathbf{c}_1) = \mathcal{Z}(\mathbf{R}\mathbf{G}_1, \mathbf{R}\mathbf{c}_1)$;
- are closed under Minkowski sum: $\mathcal{Z}(\mathbf{G}_1, \mathbf{c}_1) \oplus \mathcal{Z}(\mathbf{G}_2, \mathbf{c}_2) = \mathcal{Z}(\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}, \mathbf{c}_1 + \mathbf{c}_2)$;
- are symmetric, w.r.t. their center: $-Z_1 = -\mathcal{Z}\{\mathbf{G}_1, \mathbf{c}_1\} = \mathcal{Z}\{\mathbf{G}_1, -\mathbf{c}_1\}$
- their volume has an explicit formulation (Gover et al. 2010):
  $\mathrm{Vol}(\mathcal{Z}(G, c)) = \sum_{1 \leq j_1 < j_2 \dots j_d \leq m} \left|\det(G^{j_1 \cdots j_d})\right|$
- their corresponding half-space representation (Althoff et al. 2010):

$$\mathcal{Z}(G, c) = \bigcap_{1 \leq j_1 < \dots j_{d-1} \leq m} \{x \in \mathbb{R}^d : |h_i(x - c)| \leq k_i\},$$

$$h_i \perp g_{j_l}, \forall j_l \in \{j_1 \dots j_{d-1}\}, \quad k_i = \sum_{j_l \notin \{j_1 \cdots j_{d-1}\}} \left| h_i^\top g_{j_l} \right|$$

Back to Zonotopic Approximation.