Constrained Minimum-time-oriented Stabilization of Extended Chained Form Systems

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Second-order non-holonomic system with saturation constraints on the control inputs

$$\begin{cases} \ddot{x}_1 & = & u \\ \ddot{x}_2 & = & v \\ \ddot{x}_3 & = & x_2 u \end{cases}$$

$$-u^{max} \leq u \leq u^{max}$$

$$-v^{max} \leq v \leq v^{max}$$

Objective

Use the Receding horizon strategy to formulate the feedback law.

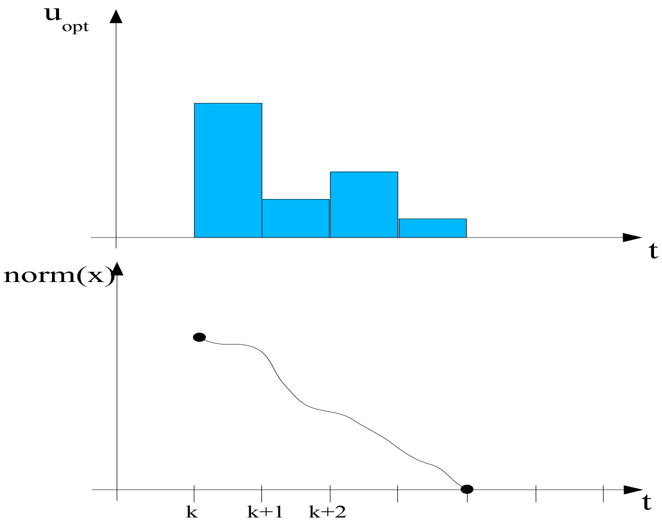
Characteristics of the extended chained form system

V/STOL system model

$$\begin{aligned}\ddot{x} &= -\sin(\theta)v_1 + \gamma\cos(\theta)v_2 \\ \ddot{y} &= \cos(\theta)v_1 + \gamma\sin(\theta)v_2 \\ \ddot{\theta} &= v_2\end{aligned}$$

- Does not satisfy the Brockett conditions.
- Contains drift component.

Murray, R.M. and Sastry, S.,"Nonholonomic motion planning: Steering using sinusoids", *IEEE Transactions on Automatic Control*, 1993, vol.38, pp 700-716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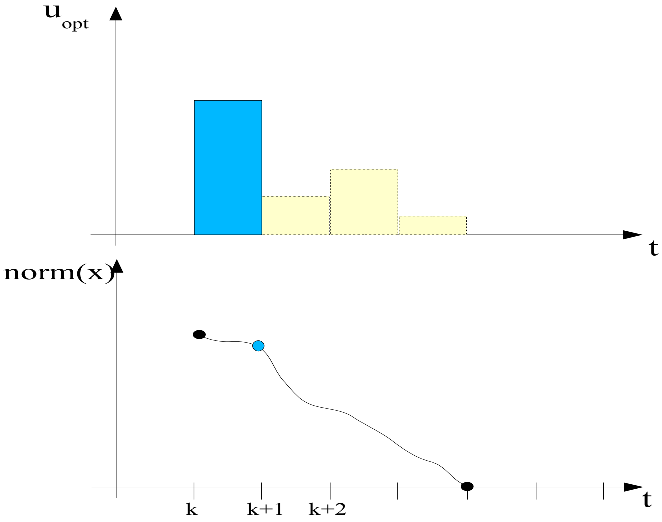
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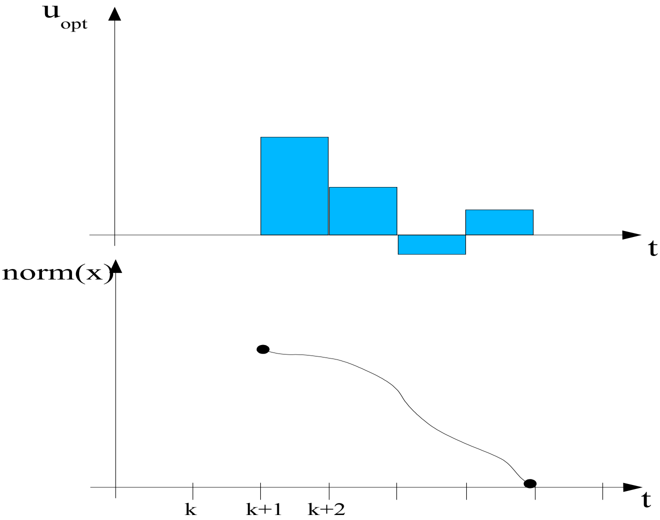
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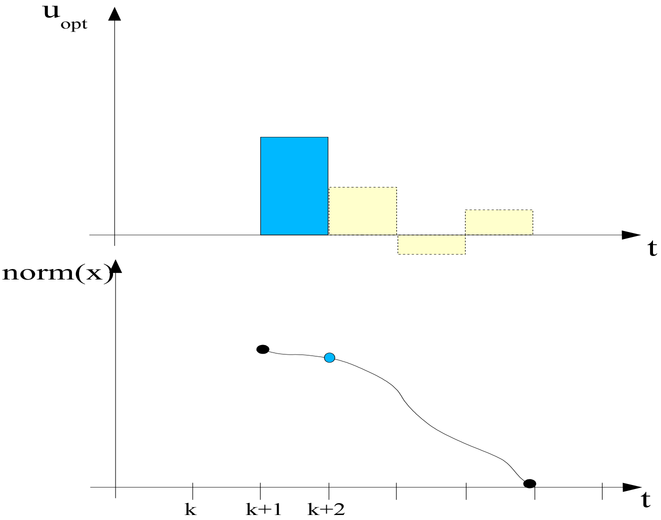
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k+1
k+2

Subsystem Σ_1

$$\begin{aligned} \dot{\zeta} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \zeta + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ \dot{\zeta} &= A_{\Sigma_1}\zeta + B_{\Sigma_1} u \end{aligned}$$

Subsystem Σ_2

$$\begin{aligned} \dot{z} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v \\ \dot{z} &= A_{\Sigma_2} z + B_{\Sigma_2} v \end{aligned}$$

$\zeta = (x_1; \dot{x}_1) \quad z = (x_3; \dot{x}_3; x_2; \dot{x}_2)$

The first control input $U = [u_0 \; u_1 \; \ldots \; u_{n-1}]^T$ such that

$$u_i = \alpha i + \beta = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

The system is steered to the desired state ζ_d in n sampling periods

$$\zeta_d = \zeta_n = A_1^n \zeta_0 + \sum_{i=0}^{n-1} (A_1)^{n-1-i} B_1 u_i$$

A_1 and B_1 : discretization of Σ_1.

Lemma

$\forall n > 0$, Γ_n *is a full rank.*

Proof.

$$\Gamma_n = \sum_{i=0}^{n-1} (A_1)^{n-1-i} B_1 \begin{bmatrix} i & 1 \end{bmatrix}$$

$$\Gamma_n = \begin{bmatrix} \frac{\delta^2 n(n-1)(n-\frac{1}{2})}{2} - \delta^2 \sum_{i=0}^{n-1} i^2 & \frac{\delta^2 n^2}{2} \\ \frac{\delta n(n-1)}{2} & \delta n \end{bmatrix}$$

$\Gamma_{21} = \frac{n-1}{2}\Gamma_{22}$ but $\Gamma_{11} \neq \frac{n-1}{2}\Gamma_{12}$ □

$$\begin{bmatrix} \alpha & \beta \end{bmatrix}^T = \Gamma_n^{-1}[\zeta_n - (A_1)^n \zeta_0]$$

The system is steered to z_d in n sampling periods

$$z_d = z_n = \Phi_n z_0 + \Psi_n V$$

The second control input V

$$V = [v_0 \; v_1 \; \dots \; v_{n-1}]^T$$

$$\begin{aligned} \Phi_n &= \prod_{j=1}^{n} A_{2_{j+1}} \\ \Psi_n &= (A_{2_{n-1}} A_{2_{n-2}} \cdots A_{2_1} B_{2_0}, A_{2_{n-1}} A_{2_{n-2}} \cdots A_{2_2} B_{2_1}, \dots A_{2_{n-1}} B_{2_{n-2}}, B_{2_{n-1}}) \end{aligned}$$

A_{2i} and B_{2i}: discretization of Σ_2 and replacing u_i by its value.

Lemma

$\forall n > 4$, Ψ_n is a full rank.

The second input

$$V = \Psi_n^+ \left(z_n - \Phi_n z_0 \right)$$

where Ψ_n^+ denotes the Moore-Penrose Pseudo inverse of Ψ_n

Lemma

$\lim_{n\to\infty} ||U|| \to 0$
$\lim_{n\to\infty} ||V|| \to 0$

Proof.

$$u_i = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\Gamma_n^{-1} \sim \frac{1}{n^4} \begin{bmatrix} n & n^2 \\ n^2 & n^3 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \Gamma_n^{-1}[\zeta_n - (A_1)^n \zeta_0]$$

$$[\zeta_n - (A_1)^n \zeta_0] \sim \begin{bmatrix} n \\ 1 \end{bmatrix}$$

$$\Gamma_n^{-1}[\zeta_n - (A_1)^n \zeta_0] \sim \frac{1}{n^4} \begin{bmatrix} n^2 \\ n^3 \end{bmatrix}$$

□

identical proof for V

Applied algorithm $A(x_0)$

1. Initial future prediction horizon n.
2. Calculate Γ_n, Φ_n and Ψ_n.
3. Compute U and V.
4. Test if U and V belong to the admissible region.
5. If the test fails, increase the prediction horizon: $n = n + 1$.
6. The algorithm is executed from step (2) until u_i and v_i, that respect the saturation constraints, are found.

$$\Sigma_2 : \dot{z} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$u_i = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \Gamma_n^{-1}[\zeta_n - (A_1)^n \zeta_0]$$

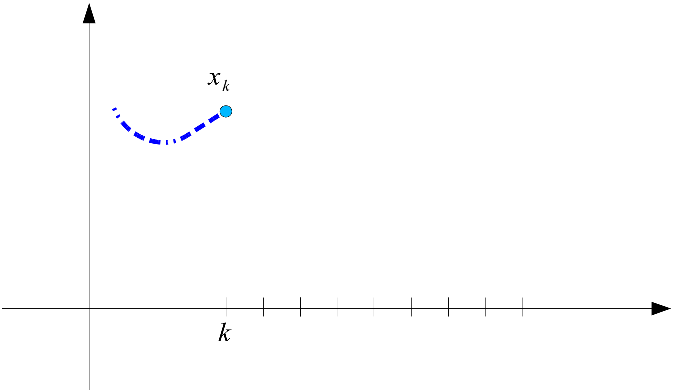
Singularity

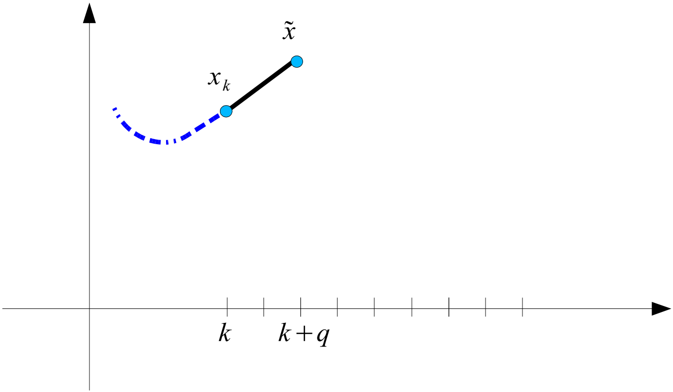
$$u = 0 \Leftrightarrow \alpha = \beta = 0 \Leftrightarrow \zeta_n = (A_1)^n \zeta_0$$

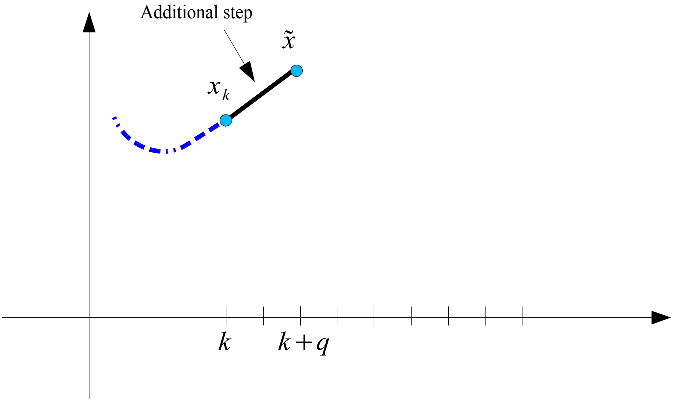
Solution: The following control is applied

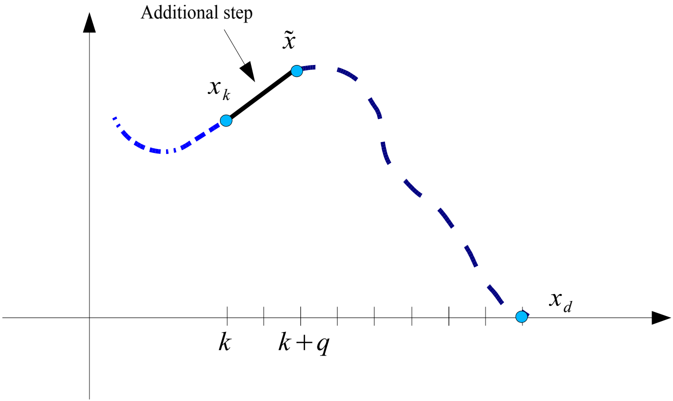
$$\begin{pmatrix} \varepsilon u^{max} \\ 0 \end{pmatrix}$$

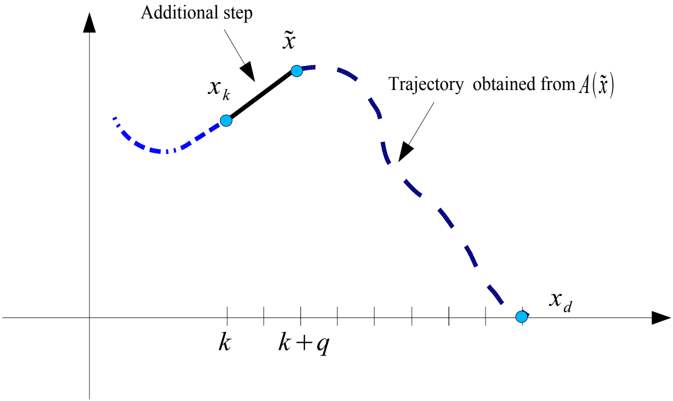
during q sampling periods and $\varepsilon \in \{-1, 1\}$. $\tilde{x}$ is reached which itself is steered to the final desired state x_d by the application of the algorithm $A(\tilde{x})$.


x_k
k


$\tilde{x}$
x_k
k
$k+q$


Additional step
$\tilde{x}$
x_k
k
$k+q$


Additional step
$\tilde{x}$
x_k
x_d
k
$k+q$


Additional step
$\tilde{x}$
x_k
Trajectory obtained from $A(\tilde{x})$
x_d
k
$k+q$

Set of trajectories

- Traj$(A(x_k))$ for $q = 0$
- Traj$(A(\tilde{x}(q, \varepsilon)))$ for $q = 1$ and $\varepsilon = 1$
- Traj$(A(\tilde{x}(q, \varepsilon)))$ for $q = 1$ and $\varepsilon = -1$
- Traj$(A(\tilde{x}(q, \varepsilon)))$ for $q = 2$ and $\varepsilon = 1$
- Traj$(A(\tilde{x}(q, \varepsilon)))$ for $q = 2$ and $\varepsilon = -1$
- ...
- Traj$(A(\tilde{x}(q, \varepsilon)))$ for $q = q_{max}$ and $\varepsilon = 1$
- Traj$(A(\tilde{x}(q, \varepsilon)))$ for $q = q_{max}$ and $\varepsilon = -1$

The shortest (in time) trajectory is chosen.
$\hat{q}$, $\hat{\varepsilon}$ and w^{opt} are obtained.

$$w^{opt}(x_k) = (w_0^{opt}, \dots, w_{\hat{q}(x_k)}^{opt}, \hat{W}(\tilde{x}))$$

$$w_j^{opt} = \begin{pmatrix} \hat{\varepsilon}(x_k)u^{max} \\ 0 \end{pmatrix} \quad \text{for } j = 1, \dots, \hat{q}(x_k)$$

$\hat{W}(\tilde{x})$ is the control obtained from the algorithm $A(\tilde{x})$.

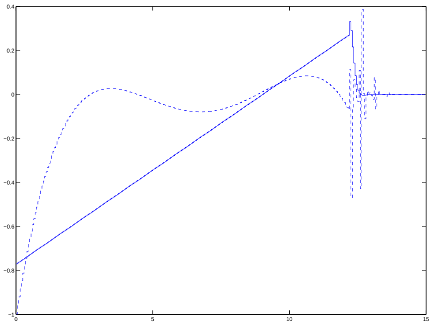
Theorem

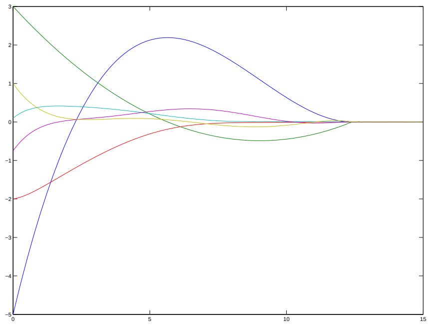
Discrete-time state feedback law defined for all $\sigma \in [0, \delta[$, δ *sampling period.*

$$w(k\delta + \sigma) = w_0^{opt}(x_k)$$

assures the global stabilization of extended chain system.

$[u^{max}, v^{max}] = [1, 1]$

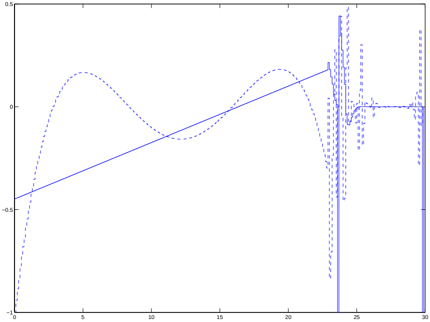


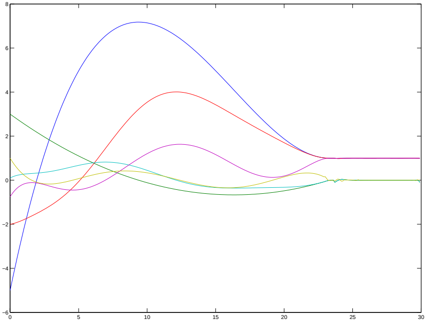


u (continuous line)
v (dashed line)

$x_0 = [-5, 3, -2, 0.1, -0.75, 1]^T$
$x_d = [0, 0, 0, 0, 0, 0]^T$

$[u^{max}, v^{max}] = [1, 1]$





u (continuous line)

v (dashed line)

$x_0 = [-5, 3, -2, 0.1, -0.75, 1]^T$

$x_d = [1, 0, 1, 0, 1, 0]^T$

Conclusion

A state feedback control

- minimum time oriented stabilization of the extended chained form system.
- respects the saturation constraints on the control inputs.
- handles the singular situations.
- fast robust computation.

Perspectives

Improved version for the PVTOL aircraft systems.