Robust Control with Classical Methods – QFT

Per-Olof Gutman

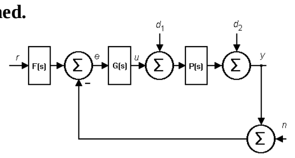
- Review of the classical Bode-Nichols control problem
- QFT in the basic Single Input Single Output (SISO) case
- Uncertainty and Fundamental Design Limitations
- QFT for non-minimum phase and computer controlled systems
- QFT for cascaded systems, and for a class of non-linear plants
- QFT for Multi-Input Multi-Output (MIMO) plants
- A comparison between QFT and other robust and adaptive control

QFT in the basic Single Input Single Output (SISO) case

- **The robust control problem**
- **Feedback**
- **The essence of QFT**
- **QFT in six steps**
- **Qsyn – *the* Toolbox for Robust Control Systems Design for use with Matlab**
- **Plant uncertainty**
- **Specifications**
- **Horowitz-Sidi bounds**
- **Feedback compensator design**
- **Prefilter design**
- **Simulations**
- **Improving the design**
- **When does QFT give a solution?**
- **Exercises**

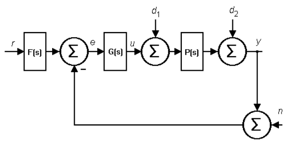
The robust control problem

- **In QFT, in general a *Canonic two degree-of-freedom structure* is assumed.**



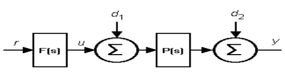
- **The robust control problem is defined as the problem to find a feedback compensator *G*(*s*) and, where appropriate, a prefilter *F*(*s*) , such that the *closed loop specifications are satisfied for all plant cases*.**

The robust control problem, cont'd



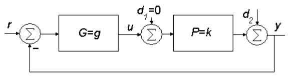
- ***G*(*s*) is used to *stabilize* the plant, *attenuate disturbances*, and *reduce the closed loop uncertainty* relative to the plant uncertainty.**
- ***F*(*s*) is used to shape the closed loop transmission from reference to output.**

Feedback



- *Open loop control*, or *feed-forward control* is possible only if
  - ***P*(*s*) is stable**; and
  - the u**ncertainty** of *P*(*s*) is acceptably small, and
  - the **un-measurable disturbances are acceptably small in relation to the specifications**.

A scalar feedback example

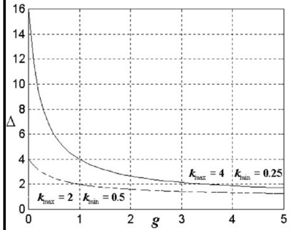


$P = k \in [k_{min}, k_{max}]$, $G(s) = g$ scalar

- Define k_{max}/k_{min} as **plant uncertainty (=** open loop uncertainty, here)
- Complementary sensitivity: $\bar{S}(s) = \dfrac{P(s)G(s)}{1+P(s)G(s)} = \dfrac{kg}{1+kg} < 1$

  $\bar{S}_{min} = \dfrac{k_{min}g}{1+k_{min}g} \to 1, \quad \bar{S}_{max} = \dfrac{k_{max}g}{1+k_{max}g} \to 1, \quad g \to \infty$
- Define $\Delta = \left|\bar{S}_{max}/\bar{S}_{min}\right|$ as **the closed loop uncertainty**
- Sensitivity: $S(s) = \dfrac{y(s)}{d_2(s)} = \dfrac{1}{1+P(s)G(s)} = \dfrac{1}{1+kg} \to 0, \quad g \to \infty$

A scalar feedback example, cont'd



- For given k_{min}/k_{max}, larger g gives smaller S and smaller Δ
- For given g, smaller k_{min}/k_{max} gives smaller Δ

Closed loop specs — **Feedback gain**
Trade-off
Plant uncertainty

In QFT, the *Trade-off* is done at each frequency

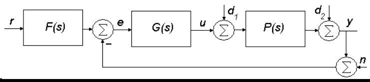
The prefilter

A prefilter shapes the closed loop transmission

$$T(s) = \frac{y(s)}{r(s)} = \frac{P(s)G(s)F(s)}{1+P(s)G(s)}$$

independently of sensitivity and uncertainty reduction at each frequency.

In the example, $F(s) = f$ can e.g. be chosen s. t. $T(s) = \frac{kgf}{1+kgf} > 1$ for all plant cases.



Cost of feedback

- **In reality, high feedback gain causes instability**
- **Noise amplification at the plant input,**

$$\frac{u(s)}{n(s)} = -\frac{G(s)}{1+P(s)G(s)}$$

may become unacceptable.

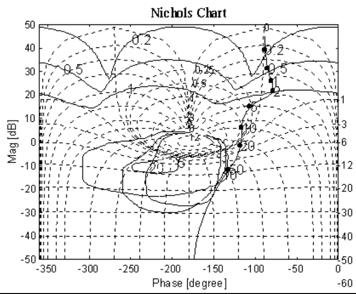
The essence of QFT

- **Conventional design:**
  solve the *simultaneous* feedback control problem for all plant cases
- **In QFT:**
  the problem is transformed to a
  - *conventional feedback design problem for one nominal plant* only,
  - with *frequency dependent constraints* on the nominal open loop,

$$L_{nom}(s) = P_{nom}(s)G(s)$$

- **In the complex plane, the constraints are called *Horowitz-Sidi bounds*.**

The Horowitz-Sidi bounds



are computed such that if the nominal open loop

$$L_{nom}(s) = P_{nom}(s)G(s)$$

satisfies them, then the closed loop satisfies the specifications with respect to uncertainty reduction, $|\bar{S}_{max}/\bar{S}_{min}|$, and disturbance attenuation for all plant cases.

QFT in six steps

1. **Determine the set of plant transfer functions $\{P_i(s)\}$. Assign one arbitrary transfer function, $P_{nom}(s)$ as the nominal.**
2. **Determine the closed loop specifications in the frequency domain. Specifications given in the time domain are translated.**
3. **Compute the Horowitz-Sidi bounds for a wisely selected set of frequencies, $\{\omega_k\}$ rad/s.**
4. **Display the nominal open loop $L_{nom}(s) = P_{nom}(s)G(s)$ in a Nichols chart, and design $G(s)$ by classical loop shaping such that $L_{nom}(s)$ satisfies the Horowitz-Sidi bounds at $\{\omega_k\}$. Use the Nyquist criterion to check stability for all plant cases.**
5. **Close the loop, and loop shape $F(s)$ such that the closed loop transfer function *P*(*s*)*G*(*s*)*F*(*s*)/(1+ *P*(*s*)*G*(*s*)) falls within its specifications.**
6. **Simulate in the frequency and time domains**

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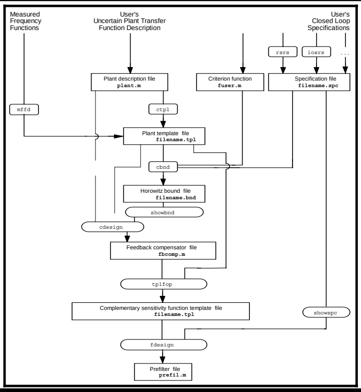
Qsyn — *the* Toolbox for Robust Control Systems Design for use with Matlab



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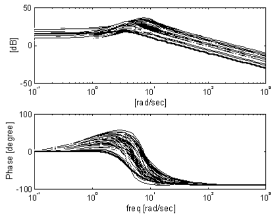
Plant uncertainty

1. Determine the set of plant transfer functions $\{P_i(s)\}$. Assign one arbitrary transfer function, $P_{nom}(s)$ as the nominal. Compute the *value sets* $\{P_i(j\omega)\}$ for the frequencies $\{\omega_k\}$

Example (Parametric uncertainty):

$$P(s)=k\cdot\frac{s+a}{1+2\varsigma s/\omega_n+s^2/\omega_n^2},$$
$$k\in[2,5], a\in[1,3], \varsigma\in[0.1,0.6], \omega_n\in[4,8]$$



- Display $P(s)$ in Qsyn: `cases('ex2_1a','all',[],1);`
- Choose arbitrary nominal: $P_{nom}(s)=k\cdot\frac{s+a}{1+2\varsigma s/\omega_n+s^2/\omega_n^2}$, $k=2,\ a=3,\ \varsigma=0.6,\ \omega_n=4$

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Plant uncertainty in Qsyn

```
function   [Par,w tpl,w nom,method,P num,P den, ...
            n dif,Uns Par] = ex2 1a

%  Plant_name :  Example 2.1. Rff method

% Definition of the parameters
% ============================

 Par = [                              %'p1=[p1min,p1max,p1nom,number of cases]'
         'k=[2, 5, 2, 8]' , ...                    % uncertain gain
         'a=[1, 3, 3, 8]', ...                     % zero: s+a
         'z=[0.3, 0.6, 0.6, 8]', ...               % complex pole
         'wn=[4, 8, 4, 8]',...
        ];

% Multiplicative unstructured uncertainty:

Uns Par=[];

% Definition of the frequency vectors [rad/sec]
% =============================================

w tpl = [0.2 0.5 1 2 5 10 20 50];   %template frequencies [rad/s]
w nom = logspace(-1,2);             %nominal frequencies  [rad/s]

% Definition of the template computation method
% =============================================

method = 'rff [1,1]';

% Plant definition: Polynomial Structure
% ======================================
%     % not used in this example, for information only  %
%     P num='k*(s+a)';                % polynomial structure,
%     P den='((s^2)/(wn^2)+2*z*s/wn + 1)'; %

% Plant definition: Real Factored Form Structure
% ==============================================

P num='(gain,k)(hf,a)';             % rff structure
P den='(dc,wn,z)';

% number of differentiators/integrators
% ======================================

n_dif = [0 0];
```


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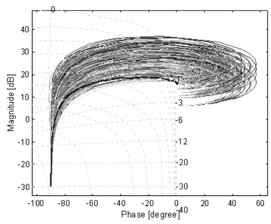
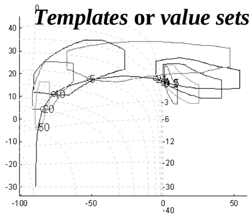
Plant uncertainty in the Nichols chart



Templates or *value sets*



`cases('ex2_1a','all',[],0);`

`ctpl('ex2_1a');`
`showtpl('ex2_1a');`

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Plant uncertainty

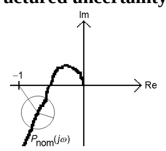
Definition: $P(s)\in\{P_i(s)\}$

Value set: $V(\omega)=\{P_i(j\omega)\}$

Parameteric uncertainty:

$P(s)\in\{P(s,q)\},\quad q\in Q\subset\Re^p$

Multiplicative un-structured uncertainty



Parametric uncertainty in Real Factored Form:

$$P(s)=\frac{ke^{-s\tau}\prod_l(1+s/b_l)\prod_m(s+b_m)\prod_q(1+2\zeta_q s/\omega_q+s^2/\omega_q^2)\prod_r(s^2+2\zeta_r\omega_r s+\omega_r^2)}{s^n\prod_u(1+s/a_u)\prod_v(s+a_v)\prod_w(1+2\zeta_w s/\omega_w+s^2/\omega_w^2)\prod_i(s^2+2\zeta_i\omega_i s+\omega_i^2)}(1+M(s))$$

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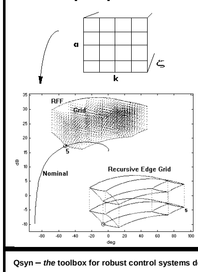
Value set computation

The grid method maps a grid in the parameter space into the complex plane:



- **Warning: The value set computed by the equidistant grid method contains 4096 points, and does not cover the whole value set.**
- **Qsyn contains better Random and Recursive Grid methods.**
- **Edge Grid methods work when the Edge Theorem holds.**
- **New Interval Analysis method**

Prudence required
- **When computing value sets**
- **When value sets not simply connected**

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Value set computation, cont'd

- Value set computation is non-trivial

$$f(x)=\frac{1}{(x-0.27)^2+0.0001}$$

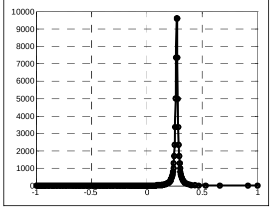
max($f(x)$)=10000

The Matlab command

```
fplot('1./((x-0.27).^2...
+0.0001)',[-1 1],'-o');
```

gives a maximim = 9615.4!
i.e. not within the stated tolerance of 0.2%.

- Known Lipschitz constant ⇒ sufficiently dense grid for prescribed accuracy.



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Value set computation, cont'd

- For functions that can be decomposed into functions whose max and min can be analytically computed over any interval, Interval Analysis methods work without explicitly computing the Lipschitz constant (Nataraj *et al*, JDMC 2003)
- Value sets of transfer functions in Real Factored form are computed within a given tolerance by concatenation of elementary value sets (Gutman *et al*, IEEE T-AC, 1994)

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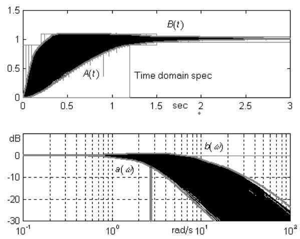
Specifications

2. Determine the closed loop specifications in the frequency domain. Specifications given in the time domain are translated to the frequency domain.

Example (Reference step response specification):

$$\begin{cases} M \le 10\% \\ t_s(5\%) \le 1.5 \text{ seconds} \end{cases}$$




```
rsrs('ex2_1a',[],[1.2 0.2],10,1.5, [], logspace(-1,2),2.85,3.1);
```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Translation of output response specification

- Specification: $l(t) \le y(t) \le u(t)$
- Model the closed loop as, e.g.,

$$C(s)=\left\{\frac{\omega_0^2}{s^2+2\varsigma\omega_0 s+\omega_0^2}\right\},\quad (\varsigma,\omega_0)\in Q$$

with

$$Q=\left\{(\varsigma,\omega_0)\,\middle|\, l(t)\le L^{-1}\left\{\frac{\omega_0^2}{s^2+2\varsigma\omega_0 s+\omega_0^2}\cdot\frac{1}{s}\right\}\le u(t)\right\}$$

- Then the "translation" is, in case of a *servo specification*,

$$a(\omega)\le\left|\frac{F(j\omega)G(j\omega)P(j\omega)}{1+G(j\omega)P(j\omega)}\right|\le b(\omega)$$

where

$$a(\omega)=\min_Q|C(j\omega)|,$$
$$b(\omega)=\max_Q|C(j\omega)|$$

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Specifications, cont'd

- The servo specification

$$a(\omega)\le\left|\frac{F(j\omega)G(j\omega)P(j\omega)}{1+G(j\omega)P(j\omega)}\right|\le b(\omega)$$

⇒ a remaining uncertainty specification for the complementary sensitivity

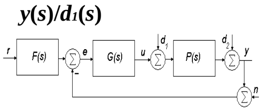
$\bar{S}(s)$ = $P(s)G(s)/(1+P(s)G(s))$, the *tolerance specification*:

$$\max\left|\bar{S}(j\omega)\right|\Big/\min\left|\bar{S}(j\omega)\right|\le b(\omega)/a(\omega)$$

- Sensitivity specification

$$\left|S(j\omega)\right|=1\Big/\left|1+L(j\omega)\right|\le x(\omega)$$

- Other specifications:
  E.g. plant input disturbance response specification
  $y(s)/d_1(s)$



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Horowitz-Sidi bounds

3. For a wisely selected set of frequencies, $\{\omega_k\}$ rad/s, compute the Horowitz-Sidi bounds from the plant value sets and from the frequency domain specifications

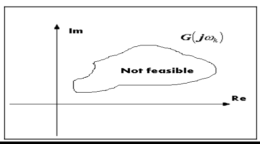
- Tolerance bounds

The tolerance specification

$$\left|\frac{\bar{S}_{\max}}{\bar{S}_{\min}}\right|=\frac{\max_i\left|P_i(j\omega_k)G(j\omega_k)/(1+P_i(j\omega_k)G(j\omega_k))\right|}{\min_i\left|P_i(j\omega_k)G(j\omega_k)/(1+P_i(j\omega_k)G(j\omega_k))\right|}\le\frac{b(\omega_k)}{a(\omega_k)}$$

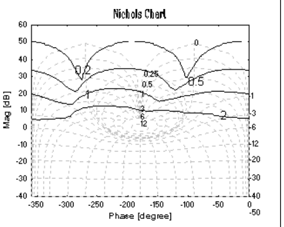
is satisfied, for each frequency ω_k, by $G(j\omega_k) \in$ feasible set. Its border $B_G(\omega_k)$ is the Horowitz-Sidi bound for $G(j\omega_k)$. Then the Horowitz-Sidi bound for $L_{\text{nom}}(\omega_k)$ is

$$B_L(\omega_k)=B_G(\omega_k)P_{\text{nom}}(j\omega_k)$$



Horowitz-Sidi bounds, cont'd

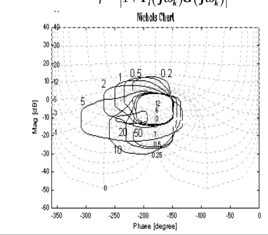
- **Tolerance bounds**




```
cbnd('ex2_1a','rsrs');
showbnd('ex2_1a',gcf,[],'rsrs')
```

- **Sensitivity bounds**

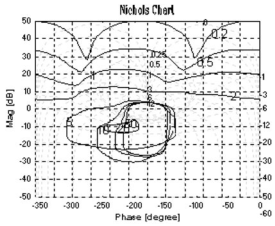
$$|S_{\max}(j\omega_k)| = \max_i \left|\frac{1}{1+P_i(j\omega_k)G(j\omega_k)}\right| \le x(\omega_k)$$



Horowitz-Sidi bounds, cont'd

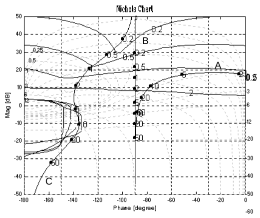
- **Other bounds**
  - **Plant input disturbance rejection bound**
  - **Noise attenuation bound**
  - **Delay margin bound**
  - **…**
- **Computations**
  - **Analytical, for some bounds**
  - **Grid the $G(j\omega_k)$-plane**

- **Dominant bounds**



Feedback compensator design

4. Display the nominal open loop $L_{nom}(s) = P_{nom}(s)G(s)$ in a Nichols chart, and design $G(s)$ by classical loop shaping such that $L_{nom}(s)$ satisfies the Horowitz-Sidi bounds at $\{\omega_k\}$. Use the Nyquist criterion to check stability for all plant cases.



- **Example, cont'd**

$$P_{nom}(s) = k\cdot\frac{s+a}{1+2\varsigma s/\omega_n + s^2/\omega_n^2},$$

$$k=2,\ a=3,\ \varsigma=0.6,\ \omega_n=4$$

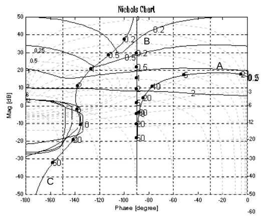
A: $G(s) = 1$

```
function [G]=Ga(s)
G = 1;
```


Feedback compensator design, cont'd

B: $G(s) = \frac{(1+2\cdot 0.6s/4+s^2/16)}{s(1+s/3)}$

```
function [G]=Gb(s)
G = (1+2*0.6*s/4+s.*s/16)/…
(s.*(1+s/3));
```




```
ha=cdesign('ex2_1a','Gb');
```

C: $G(s) = \frac{2.5(1+s/6)(1+2\cdot 0.6s/4+s^2/16)}{s(1+s)(1+s/3.2)(1+s/26)}$

```
function [G]=Gc(s)
num= 2.5*(1+s/6).*…
      (1+2*0.6*s/4+s.*s/16);
den= s.*(1+s).*…
(1+s/32).*(1+s/26);
G = num./den;
```


Asymptotic closed loop stability

If

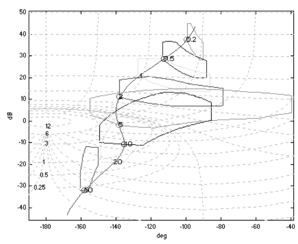
- **the plants are rational (with or without delay),**
- **have common high frequency gain sign,**
- **have simply connected value sets in the extended Nichols chart, and**
- **the compensated nominal $L_{nom}(s)$ satisfies the Nyquist stability theorem,**
- **$L_{nom}(s)$ satisfies a Horowitz sensitivity bound,**

then

♠ **the Nyquist stability theorem is satisfied for each plant case.**

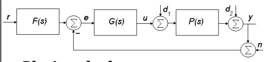
Asymptotic closed loop stability, cont'd

- **The compensated open loop templates**



Prefilter design

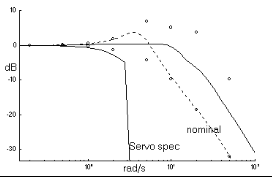
5. Close the loop, and loop shape $F(s)$ such that the closed loop transfer function $P(s)G(s)F(s)/(1+ P(s)G(s))$ falls within its specifications.



- **Closing the loop: compute value sets of the complementary sensitivity function**

$$\bar{S}(s) = \frac{P(s)G(s)}{1+P(s)G(s)}$$

```
tplfop('cex2_1a','iosrs',…
       [],'ex2_1a',1,'Gc');
showspc(''ex2_1a','rsrs',…
                  'freq');
fdesign(''cex2_1a.tpl');
```



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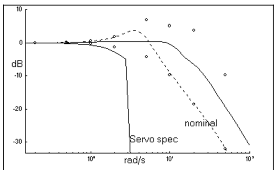
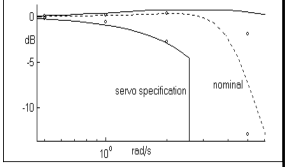
Prefilter design, cont'd



Notice: the gain extent of $\bar{S}$ is less or equal to the allowed closed loop gain extent in the servo specification, if $L_{nom}(j\omega)$ was designed to satisfy the tolerance bounds.

$$F(s) = \frac{1}{\left(1 + 2\cdot 0.83s/3.4 + s^2/3.4^2\right)}$$

```
fdesign('cex2_1a.tpl','Fa');
```



Qsyn – *the* toolbox for robust control systems design P-O Gutman

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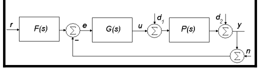
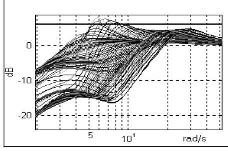
Simulations

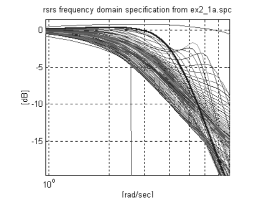
6. Simulate in the frequency and time domains.



- **Sensitivity**



- **Closed loop frequency functions**




```
ccases('ex2_1a',par,'rsrs',…
'Gc','Fa',…
logspace(-1,2,120),'mag');
```

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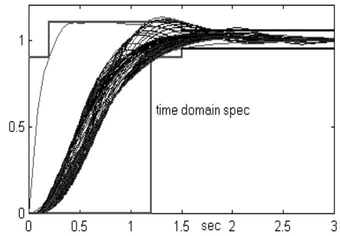
Simulations, cont'd

- **Step responses**



What went wrong?

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Improving the design

- **When the specifications are not satisfied ...**
  - **include more frequencies in $\{\omega_k\}$**
  - **take account of the approximate nature of the time domain-to-frequency domain "translation"**
  - **avoid "clever" loop shaping.**
- **When there is severe "over-design"...**
  - **re-design $G(s)$ such that $L_{nom}(s)$ resides closer to the Horowitz-Sidi bounds $\Rightarrow$ lowering the band-width**
  - **if necessary, re-design $F(s)$**

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When does QFT give a solution?

If the *minimum phase* plants $P_i(s)$ are such that

- **they have common high frequency gain sign,**
- $\lim_{s\to\infty} P_i(s) = k_i / s^{d_i},\ k_i \in [\underline{k}, \bar{k}],\ d_i \in [\underline{d}, \underline{d}+1, \ldots, \bar{d}-1, \bar{d}]$ **for all i**

then

- **any tolerance specification, and**
- **any sensitivity specification > 0 dB**

is achievable with a strictly proper $G_i(s)$

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Exercises

1. **Solve the example problem!**
2. **Solve the example problem with**
   a) a plant input disturbance rejection specification, in order to avoid the present inverse design in which $P_{nom}^{-1}(s)$ is a factor of $G(s)$, *and/or*
   b) a delay, $e^{-0.010s}$, introduced into the plant, in order to make the plant NMP, and avoid infinite-bandwidth solulutions.
3. **Solve the robust control problem,**

$$P(s) = \frac{k}{s^2} e^{-\tau s}, k = \{1,10\}, \tau = \{0,1\}.$$

with the specification |S|<6dB, and ω_c >0.07 rad/s.