Some thoughts about output feedback and robustness in Model Predictive Control

Per-Olof Gutman

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From LP-OLOF to MPC

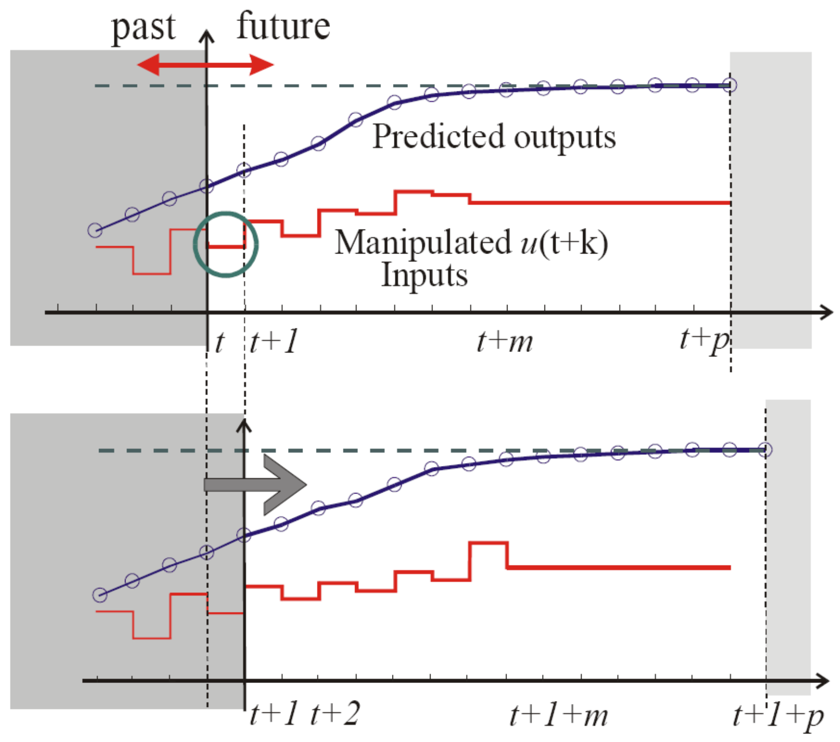
Per-Olof Gutman

Abstract: Model Predictive Control has become a widely used control method in industry for plants for which *i*) a reasonably accurate model is available, *ii*) control and/or state constraints are essential specifications, and *iii*) either the control computer power is sufficient to solve an optimization problem within one sampling interval, or the optimal control signal can be pre-computed and stored and used in a convenient way. Some issues such as feasibility and stability are reviewed in the light of LP-OLOF, and some un-resolved topics such as output feedback and robustness are discussed.

The MPC idea

- What is wrong with conventional linear or non-linear control?
  - Control and state constraints.
  - Anti-reset windup gives only a partial solution.
- What is wrong with optimal control?
  - It is open-loop in the usual absense of a closed-form solution.

- MPC: optimal+feedback+constraints



- Optimize at time t (new measurements)
- Only apply the first optimal move $u(t)$
- Repeat the whole optimization at time $t+1$

History

- Zadeh (1962), Propoi (1963): minimum time problem for linear discrete-time systems with control constraints can be seen as a series of LP problems.
- Propoi (1963): proposed to solve an LP problem in each sampling interval.
- Dreyfus (1964) named it **O**pen **L**oop **O**ptimal – **F**eedback.
- Cutler and Ramaker (1979) at Shell 1973ff: DMC with linear step response models, quadratic performance index, set point control.
- Knudsen (1975) LP-OLOF with control constraints on chemical pilot plant.
- Richalet et al (1976ff): MPC with linear impulse response model, quadratic cost, input/output constraints, iterative optimization.
- Gutman (1982): LP-OLOF with state and control constraints, boundary layer penalty, horizon iteration, terminal constraint, and observation that control can be pre-computed.
- Gutman and Hagander (1985): design of constrained controllers for linear systems
- Gutman (1986): water level control of the Bergeforsen dam.
- Gutman and Cwikel (1986): nec&suff stabilizability conditions for discrete time control with polyhedral state and control constraints.
- Keerthi and Gilbert (1988): MPC stability proof by terminal constraint and value function as Lyapunov function
- Blanchini (1992ff): Extension of the Gutman-Cwikel proof to the uncertain plant case.
- Bemporad, Morari, Borrelli (2002ff) Explicit, precomputed, MPC.
- Mayne et al (2000): ”Constrained model predictive control: Stability and optimality”
- Qin and Badgwell (2003): ”A survey of industrial model predictive control technology”

Stability

$$x_{k+1} = f_k(x_k, u_k), \quad y_k = g_k(x_k, u_k), \qquad k \geq i, \qquad (1)$$

$$(x_k, u_k) \in Z_k \subset R^n \times R^m, \qquad k \geq i, \qquad x_i = a \text{ is a feasible i.c.} \qquad (2)$$

Cost function: $$J_i = \sum_{k=i}^{i+H} h_k(y_k, u_k)$$

Terminal constraint: $x_{i+H} = 0$ (later relaxed to $x_{i+H} \in X$)

Keerthi and Gilbert (1988) showed stability of *moving horizon control* for a sufficiently large *H* by using J_i as a Lyapunov function.

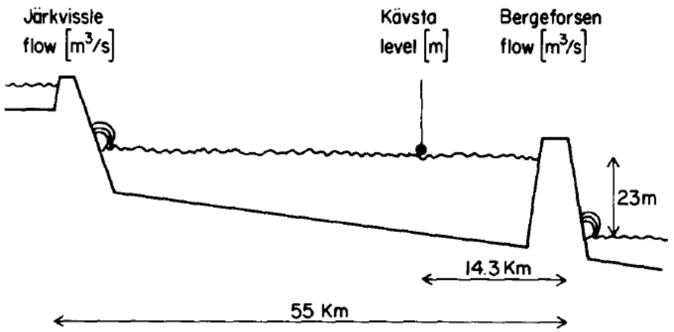
Remark: Gutman's Ph.D thesis 1982 did not contain a proof, although another paper in the thesis, the widely quoted Gutman and Hagander (1985), used control generating Lyapunov functions for constrained controller design …

LP-OLOF in practice

Automatica, Vol. 22, No. 5, pp. 533–541, 1986.

A Linear Programming Regulator Applied to Hydroelectric Reservoir Level Control*

PER-OLOF GUTMAN†



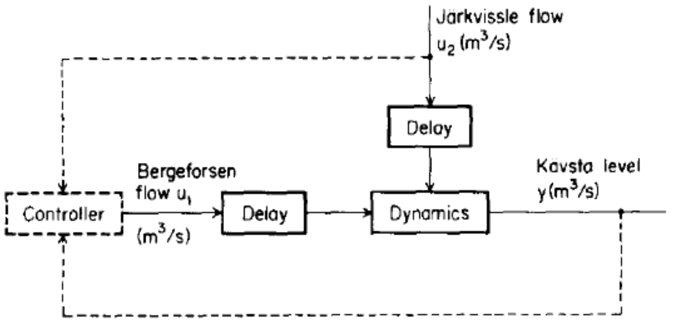


FIG. 6. Block diagram of the Bergeforsen reservoir level control

$$x(t+1) = \phi x(t) + \Gamma u_1(t) + F(t)$$

$$x(t) = (y(t)\,u_1(t-1)\,u_1(t-2)\quad u_1(t-3)\,u_1(t-4)\,u_1(t-5)\quad u_1(t-6)\,\Delta u_1(t-2)\,\Delta u_1(t-2))^{\mathrm{T}}$$

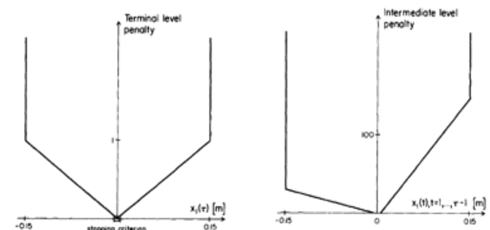


FIG. 8(b). Penalty function components: intermediate

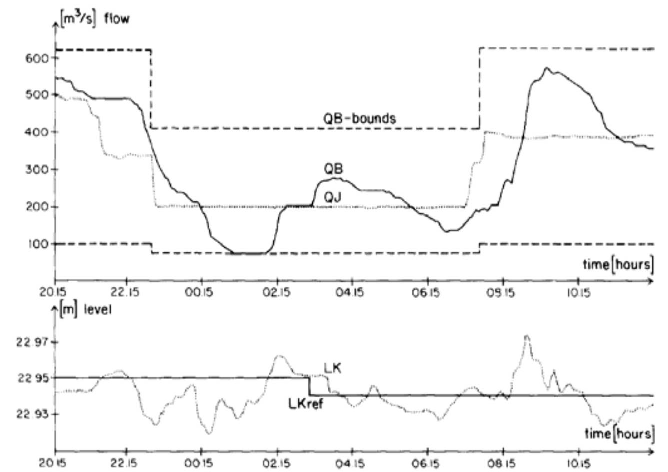


FIG. 9. Diagram of final experiment: Bergeforsen flow (QB), Järkvissle flow (QJ), Kävsta level (LK) and Kävsta reference level (LKref).

State of the art I

Implicit MPC

- In each sampling interval is the optimal control signal sequence computed over some *control horizon* H_c, such that
- the state reaches a *terminal set X* from which stabilizability is assured by other means, and
- the state trajectory is optimized over a (possibly longer) *prediction horizon* H_p, whereby
- the control over H_p - H_c, is either open loop, e.g. u=0, or given by some feedback law.
- Also for hybrid and non-linear plants.

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Feasibility

Gutman and Cwikel (1986):

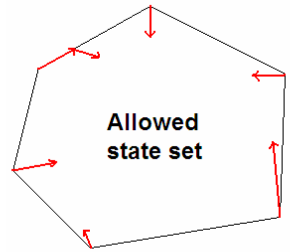
- the necessary and sufficiant condition for stabilizing a *linear* discrete time control system with *polyhedral* state and control constraints is that, at each vertex *i* of the allowed state polyhedron *X*, there exists a feasible control signal (sequence) $u_i \in U$ that brings the state to *int*(*X*).
- A stabilizing controller is given by the *convex combination of vertex controls* in each sector, with a Lyapunov function given by shrunken images of the border of *X*.

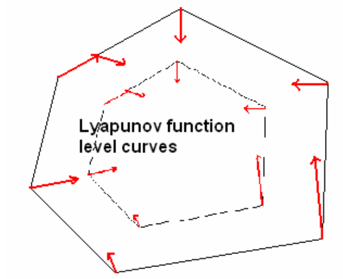
Blachini (1992ff):

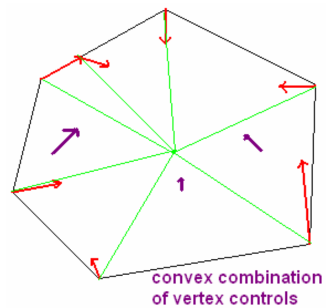
- extension to the uncertain plant case.

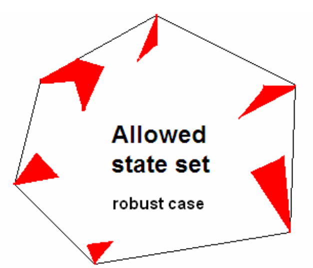
Cwikel/Gutman (1986, 1987):

- an algorithm to find maximal state constraint sets, and its convergence.









$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(t)$$

$$\Omega = \{u \mid -1 \le u \le 1\}, \quad G = \{x \mid -25 \le x_1 \le 25 \text{ and } -5 \le x_2 \le 5\}$$

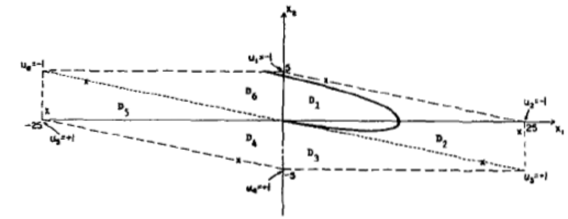


Fig. 1. X, an Ω-invariant set w.r.t. $G(---)$; the vertex controls u_i; $\Phi v_i + \Gamma u_i$, $i = 1, \cdots, 6$, where $\{v_i\}$ are the vertices of X; the subsets D_i, $i = 1, \cdots, 6$; phase portrait $x_2(x_1)$ for the system (3.5), (3.10) with $x(0) = (-2, 5)^T$.

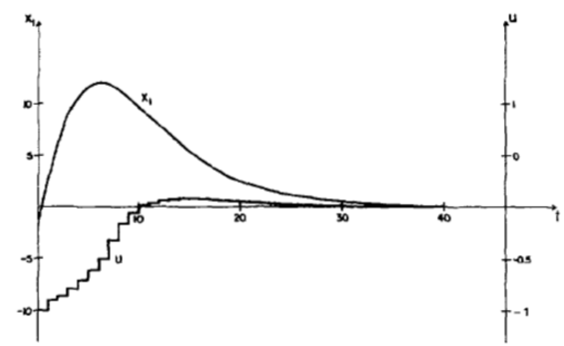


Fig. 2. $x_1(t)$ and $u(t)$ for the system (3.5), (3.10) with $x(0) = (-2, 5)^T$.

Similar formulas can be obtained in D_i, $i = 2, \cdots, 6$. The complete control law is given by:

$$u = \begin{cases} (-0.04 - 0.2)x & x \in D_1 \cup D_4 \\ (-0.04 - 0.4)x & x \in D_2 \cup D_5 \\ (0 \quad -0.2)x & x \in D_3 \cup D_6. \end{cases} \qquad (3.9)$$

State of the art II

Explicit MPC (Morari, Bemporat, Borrelli *et al*, 2002 ff)

- Feasible polyhedron X of initial states computed *off-line* by multi-parametric programming (e.g. mp-LP) where the parameterization is taken w r t the initial state $x(0)$.
- X is divided in sub-polyhedra, and an affine control law is found, *off-line* , for each subpolyhedron, $u_i = K_i x + c_i$, that is continuous over subpolyhedra boundaries.
- the control is implemented on-line by finding to which sub-polyhedron the measured state belongs
- extended to presense of bounded disturbances, and uncertain convexely combined plant matrices.

Example: Double Integrator

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

performance measure

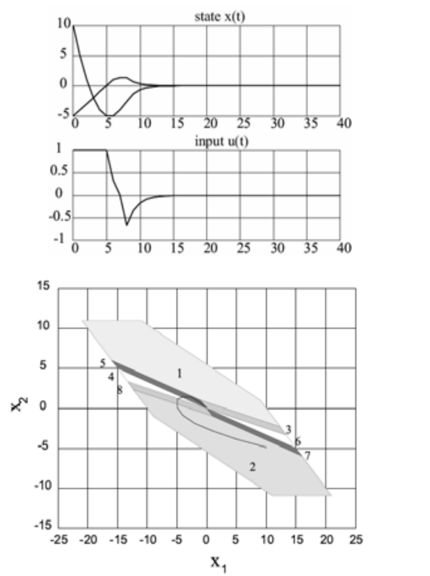
$$\sum_{k=0}^{1} \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{t+k+1|t} \right\|_\infty + |0.8 u_{t+k}|$$

input constraints

$$-1 \leq u_{t+k} \leq 1, \ \ k = 0, 1$$

state constraints

$$-10 \leq x_{t+k|t} \leq 10, \ \ k = 1, 2.$$



$$u = \begin{cases} -1.00 & \text{if } \begin{bmatrix} 1.00 & 2.00 \\ 0.00 & 1.00 \\ -1.00 & -1.00 \\ -0.80 & -3.20 \\ 1.00 & 1.00 \\ -1.00 & -3.00 \end{bmatrix} x \leq \begin{bmatrix} 11.00 \\ 11.00 \\ 10.00 \\ -2.40 \\ 10.00 \\ -2.00 \end{bmatrix} \quad \text{(Region \#1)} \\ 1.00 & \text{if } \begin{bmatrix} 0.80 & 3.20 \\ -1.00 & -2.00 \\ -1.00 & -1.00 \\ 1.00 & 1.00 \\ 0.00 & -1.00 \\ 1.00 & 3.00 \end{bmatrix} x \leq \begin{bmatrix} -2.40 \\ 11.00 \\ 10.00 \\ 10.00 \\ 11.00 \\ -2.00 \end{bmatrix} \quad \text{(Region \#2)} \\ \begin{bmatrix} -0.33 & -1.33 \end{bmatrix} x & \text{if } \begin{bmatrix} 0.53 & 2.13 \\ 0.67 & 0.67 \\ -1.00 & -1.00 \\ -0.33 & -1.33 \end{bmatrix} x \leq \begin{bmatrix} 0.00 \\ 0.00 \\ 10.00 \\ 1.00 \end{bmatrix} \quad \text{(Region \#3)} \\ 0 & \text{if } \begin{bmatrix} -0.80 & -3.20 \\ 1.00 & 3.00 \\ -1.00 & -1.00 \end{bmatrix} x \leq \begin{bmatrix} 0.00 \\ 0.00 \\ 10.00 \end{bmatrix} \quad \text{(Region \#4)} \\ \begin{bmatrix} -0.50 & -1.50 \end{bmatrix} x & \text{if } \begin{bmatrix} -1.00 & -1.00 \\ 0.50 & 0.50 \\ -0.80 & -2.40 \\ 0.50 & 1.50 \end{bmatrix} x \leq \begin{bmatrix} 10.00 \\ 0.00 \\ 0.00 \\ 1.00 \end{bmatrix} \quad \text{(Region \#5)} \\ 0 & \text{if } \begin{bmatrix} 0.80 & 3.20 \\ -1.00 & -3.00 \\ 1.00 & 1.00 \end{bmatrix} x \leq \begin{bmatrix} 0.00 \\ 0.00 \\ 10.00 \end{bmatrix} \quad \text{(Region \#6)} \\ \begin{bmatrix} -0.50 & -1.50 \end{bmatrix} x & \text{if } \begin{bmatrix} 1.00 & 1.00 \\ -0.50 & -0.50 \\ 0.80 & 2.40 \\ -0.50 & -1.50 \end{bmatrix} x \leq \begin{bmatrix} 10.00 \\ 0.00 \\ 0.00 \\ 1.00 \end{bmatrix} \quad \text{(Region \#7)} \\ \begin{bmatrix} -0.33 & -1.33 \end{bmatrix} x & \text{if } \begin{bmatrix} -0.53 & -2.13 \\ -0.67 & -0.67 \\ 1.00 & 1.00 \\ 0.33 & 1.33 \end{bmatrix} x \leq \begin{bmatrix} 0.00 \\ 0.00 \\ 10.00 \\ 1.00 \end{bmatrix} \quad \text{(Region \#8)} \end{cases}$$

Open MPC problems?

State of the art

- **Disturbance rejection**: Additive bounded disturbances can be handled in a worst case sense by either the inclusion of a forbidden *boundary layer* in the feasible state set, or directly by the optimization algorithm.
- **Robustness to plant uncertainty**. Can conservatively be ”translated” to additive disturbance (Kothare et al, 1996). Plant uncertainty defined as the system matrices being convex combinations of given matrices can also be handled in a conservative way.
- **Output feedback**. A non-minimal state representation $x = [u^T\ y^T]^T$ with input and output constaints has been occasionally used.
- **Explicit MPC**. At present used only for low-order plants

Open problems, maybe ...

- **Output feedback**. What is the price of using a non-minimal state representation in the MPC setting ? Non-unique optimal solutions?
- **Robustness to plant uncertainty**, in particular in the output feedback setting. Can parametric uncertainty be handled without excessive conservatism?

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