Polytopic methods, set invariance and predictive control for different classes of systems with variable time-delay

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Outline

Outline

Introduction

Time-delay systems (Niculescu 2001)**:**

- Complicated time behavior:
  - Oscillations
  - Instabilities
- Infinite dimension in continuous time

Present in:

- Propagation and transport phenomena
- Population dynamics (reproduction, development or extinction)
- Economic systems:
  - Investment policy
  - Market evolution
  - Analysis and decision
- Communication $\rightarrow$ *non-zero time interval* between the initiation and delivery of a signal.

Introduction

Objectives:

- Constrained control for LTI systems with variable and bounded time-delay

Tools:

- Time-delay system $\rightarrow$ Polytopic uncertain system
    - Embeddings $\rightarrow$ Convex bodies and convex geometry
    - Classical robust stabilization
- Lyapunov-Krasovskii candidate

Constrained control:

- Construct robust positive invariant sets
- Design constrained predictive control laws

Problem Formulation

- Continuous linear system with input delay:

$$\dot{x}(t) = A_c x(t) + B_c u(t-h)$$

- Degree of uncertainty: $h = dT_e - \epsilon$, with sampling time T_e.
- Discrete model:

$$x_{k+1} = Ax_k + Bu_{k-d} - \Delta(u_{k-d} - u_{k-d+1})$$

$$u(t) = u_k, \ \forall \ t \in [t_k, t_{k+1})$$

where:

$$A = e^{A_c T_e}, \quad B = \int_0^{T_e} e^{A_c(T_e-\theta)} B_c d\theta, \quad \Delta = \int_{-|\epsilon|}^{0} e^{-A_c\tau} B_c d\tau$$

- $\Delta \rightarrow$ exponential function in terms of the uncertainty ϵ.

Objectives

Objective:

- **Robust stability of LTI systems with time-variable delay**
- **Design a control law which regulates the system state while robustly satisfying a set of constraints:**

$$Cx_k + Du_k \leq W$$

Outline

Polytopic Embeddings

Goals:

- **Confine Δ in a polytope** $\rightarrow$ convex set covering all the possible realizations
- Basic element of construction: Jordan canonical form $A_c = V \Lambda V^{-1}$
- Low complexity polytopes $\rightarrow$ Simplex ($n+1$ realizations)

Cases to be treated:

1. Non defective matrix A_c with real and non repeated eigenvalues
2. Non defective matrix A_c with complex conjugated eigenvalues
3. Defective matrix A_c with real and repeated eigenvalues

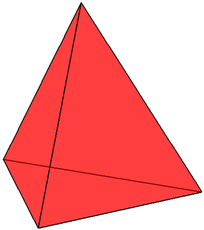
Simplex

Definition

Simplex or n-simplex: Is the convex hull of a set of $(n+1)$ affinely independent points in an Euclidean Space of n dimension.

Why the choice of a **simplex**?

- Reduced complexity in terms of extreme points
- "Volume" expressed by an analytical function



Non defective transfer matrix with real and non repeated eigenvalues

Example: Confining Δ.

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t-h)$$

$$\epsilon = 0.1$$

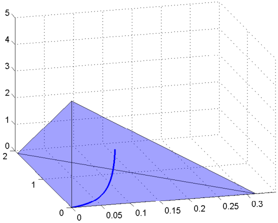
Extreme realizations:

$$\Delta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Delta_1 = \begin{bmatrix} 4.02 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 0 \\ 1.9 \\ 0 \end{bmatrix} \quad \Delta_3 = \begin{bmatrix} 0.32 \\ 0 \\ 0 \end{bmatrix}$$

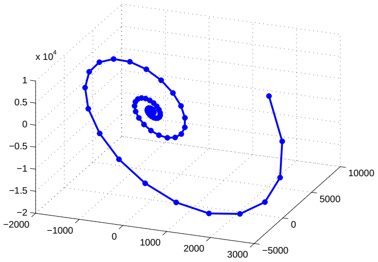
Using the result of Olaru et al. (2008):

$$\Delta_0 = \mathbf{0}_{n \times m};$$
$$\Delta_i = n\ V \int_0^{\bar{\epsilon}} e^{\Lambda_i \tau} d\tau V^{-1} B_c$$



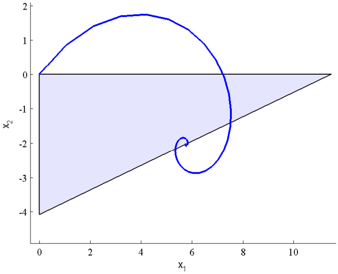
Non defective transfer matrix with complex conjugated eigenvalues

- Nonlinear dependence of $\Delta \rightarrow$ spiral type function
- Containment by a simplex is a hard task



Non defective transfer matrix with complex conjugated eigenvalues

- Nonlinear behavior of $\Delta \rightarrow$ spiral type
- Containment by a simplex is a hard task
- Using the existent techniques (Olaru et al. (2008)):



Non defective transfer matrix with complex conjugated eigenvalues

- By the Jordan decomposition:

$$\Delta = \int_{-|\epsilon|}^{0} e^{-A_c\tau} B_c d\tau = \underbrace{V\Lambda^{-1}e^{\Lambda\bar{\epsilon}}V^{-1}Bc}_{\theta} - V\Lambda^{-1}V^{-1}B_c$$

- **Extreme realizations**: Defining the center as $c_p = A_c^{-1}B_c$.
  1. A_c stable - Hypercube by forward dynamics $\dot{\theta} = \Lambda\theta$:

$$|\theta| \leq \left|V\Lambda^{-1}\right| \left|V^{-1}B_c\right|$$

  2. A_c instable - ϵ will "travel" along $[0, \bar{\epsilon}]$ from $\bar{\epsilon}$ to 0. Δ in the reverse time by $-A_c$ stable.

$$|\theta| \leq \left|V\Lambda^{-1}\right| e^{\Lambda\bar{\epsilon}} \left|V^{-1}B_c\right|$$

Hypercube constraints done by the box:

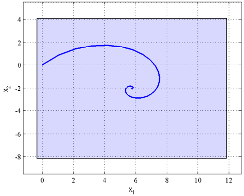
$$-|\theta| - c_p \leq \Delta_i \leq |\theta| + c_p$$

Non defective transfer matrix with complex conjugated eigenvalues

Example 1: A_c stable

$$\dot{x}(t) = \begin{bmatrix} -0.1 & 0.21 \\ -0.21 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-h)$$

$$\epsilon = 100; \quad \lambda_{1,2} = \begin{bmatrix} -0.1 + 0.21\ i \\ -0.1 - 0.21\ i \end{bmatrix}$$



Applying the results:

$$\begin{cases} c_p = A_c^{-1} B_c = \begin{bmatrix} 5.7301 \\ -2.0333 \end{bmatrix} \\ \left|V \Lambda^{-1}\right| \left|V^{-1} B_c\right| = \begin{bmatrix} 6.0802 \\ 6.0802 \end{bmatrix} \end{cases}$$

Box:

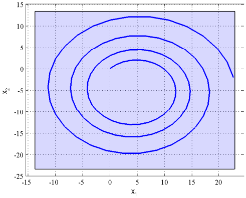
$$\begin{bmatrix} -0.3500 \\ -8.1135 \end{bmatrix} \leq |\theta| \leq \begin{bmatrix} 11.8103 \\ 4.0469 \end{bmatrix}$$

Non defective transfer matrix with complex conjugated eigenvalues

Example 2: A_c unstable

$$\dot{x}(t) = \begin{bmatrix} 0.1 & 0.21 \\ -0.21 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-h)$$

$$\epsilon = 100; \quad \lambda_{1,2} = \begin{bmatrix} 0.01 + 0.21\ i \\ 0.01 - 0.21\ i \end{bmatrix}$$



Applying the results:

$$\begin{cases} c_p = A_c^{-1} B_c = \begin{bmatrix} 4.5249 \\ -4.9774 \end{bmatrix} \\ \left|V\Lambda^{-1}\right| e^{\Lambda\bar{\epsilon}} \left|V^{-1} B_c\right| = \begin{bmatrix} 18.2851 \\ 18.2851 \end{bmatrix} \end{cases}$$

Box:

$$\begin{bmatrix} -13.7603 \\ -23.2625 \end{bmatrix} \leq |\theta| \leq \begin{bmatrix} 22.8100 \\ 13.3078 \end{bmatrix}$$

Defective transfer matrix with real eigenvalues

- Jordan decomposition $A_c = V\Sigma V^{-1}$ is the main ingredient.
- Existence of Jordan Blocks of multiplicity greater than 1.
- **Eigenvectors with linear dependence.**

Defective transfer matrix with real eigenvalues

- By the Jordan decomposition $A_c = V\Sigma V^{-1}$, and block diagonal matrix $\Sigma \in \mathbb{R}^{n\times n}$

$$\Sigma = \begin{bmatrix} \Sigma_{1,m_1} & 0 & \cdots & 0 \\ 0 & \Sigma_{2,m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{p,m_p} \end{bmatrix}; \quad \forall i \in [1, ..., p]$$

- Jordan blocks $\Sigma_{i,m_i} \in \mathbb{R}^{m_i\times m_i}$ with:

$$\Sigma_{i,m_i} = \begin{bmatrix} \sigma_i & 1 & \cdots & 0 & 0 \\ 0 & \sigma_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i & 1 \\ 0 & 0 & \cdots & 0 & \sigma_i \end{bmatrix} = \sigma_i \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{\Lambda_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Gamma_i}$$

Defective transfer matrix with real eigenvalues

- Explicit matrices:

$$\Sigma = \begin{bmatrix} \sigma_1\Lambda_1+\Gamma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2\Lambda_2+\Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p\Lambda_p+\Gamma_p \end{bmatrix} =$$

$$= \sum_{i=1}^{p} \sigma_i \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Lambda_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{L_i} + \sum_{i=1}^{p} \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Sigma_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{G_i}$$

- Compact form:

$$\Sigma = \sum_{i=1}^{p} \sigma_i L_i + G_i$$

- Contain Δ done by:

$$\Delta(\epsilon) = \int_0^{\epsilon} e^{A_c\tau} B_c d\tau = \int_0^{\epsilon} e^{V\Sigma V^{-1}\tau} d\tau \ B_c = V \int_0^{\epsilon} e^{\Sigma\tau} d\tau \ V^{-1} B_c$$

Defective transfer matrix with real and repeated eigenvalues

- Take a closer look in the exponential term.
- Exploiting the structure of the Jordan blocks:

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} e^{\Gamma_i \tau}$$

- Taylor expantion of $e^{\Sigma_{i,m_i}\tau}$ up to the $(m_i - 1)^{th}$ term:

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau}\left(I + \frac{1}{1!}\Gamma_i \tau + \frac{1}{2!}\Gamma_i^2 \tau^2 + \ldots + \frac{1}{(m_i - 1)!}\Gamma_j^{m_i - 1}\tau^{m_i - 1}\right)$$

Defective transfer matrix with real and repeated eigenvalues

Extreme realizations: $\rightarrow$ Two cases:

1. Jordan blocks of size 1. Techniques of (Olaru et al. 2008):

$$\Delta_{j,1}(\epsilon) = V \int_0^{\epsilon} e^{\Sigma\tau} d\tau \ V^{-1} B_c = V L_j \int_0^{\epsilon} e^{\sigma_j \tau} d\tau \ V^{-1} B_c$$

2. Jordan blocks of size ≥ 2. Embedding:

$$\begin{aligned}
&\Delta_j(\epsilon) = V L_j \int_0^{\epsilon} e^{\sigma_j \tau} d\tau \ V^{-1} B_c \\
&\Delta_{j,1}(\epsilon) = V \tfrac{1}{1!} G_j \int_0^{\epsilon} \tau e^{\sigma_j \tau} d\tau \ V^{-1} B_c \\
&\vdots \\
&\Delta_{j,m_j-1}(\epsilon) = V \tfrac{1}{(m_j-1)!} G_j^{m_j-1} \int_0^{\epsilon} \tau^{m_j-1} e^{\sigma_j \tau} d\tau \ V^{-1} B_c
\end{aligned}$$

Toward less conservative embeddings

- **Conservative approach** $\rightarrow$ reduce conservatism.
- Low complexity embeddings: **Hypercubes** $\rightarrow$ **Simplices**.
- Local embeddings can be constructed upon the splitting of Δ.

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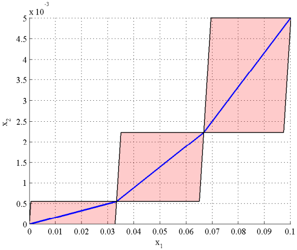


Toward less conservative embeddings

- Minimization of the *"volume"* of the simplex.

Theorem

The volume of a simplex S described by its bounding hyperplanes is given by:

$$Vol(S) = \frac{|det(H)|^n}{n!\prod_{i=0}^{n} H_{i0}}$$

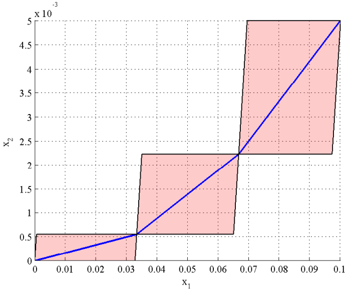
where H_{i0} is the cofactor of h_{i0} in H.

Proof: See (Gritzmann 1994) for details.

- Cost function taking into account the *"size"* of the convex hull.
- **Iterative algorithm to minimize the simplex volume at each step.**

How the algorithm works

Result: Simplex volume minimized using the vertices of the local embeddings as constraints.



Minimizing the volume!

How the algorithm works

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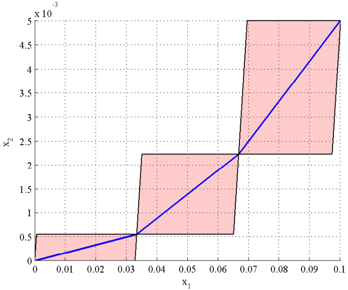
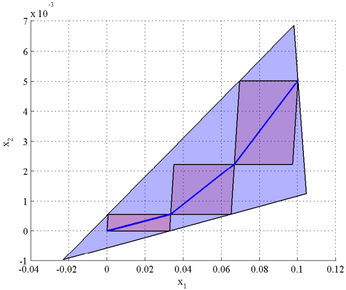
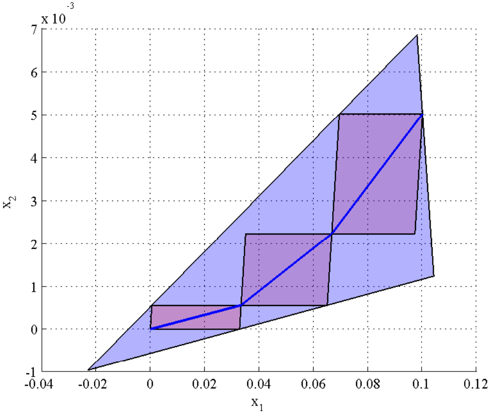




How the algorithm works

Which local embedding have the biggest number of vertices on the frontier of the polytopic embedding?



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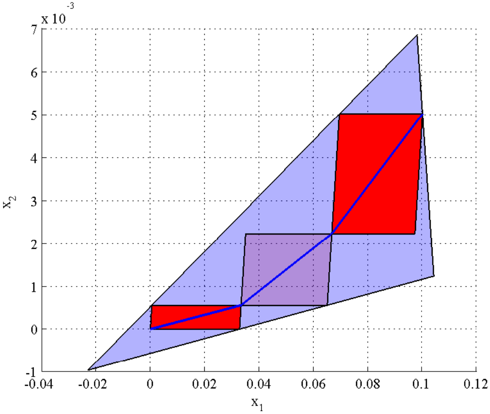
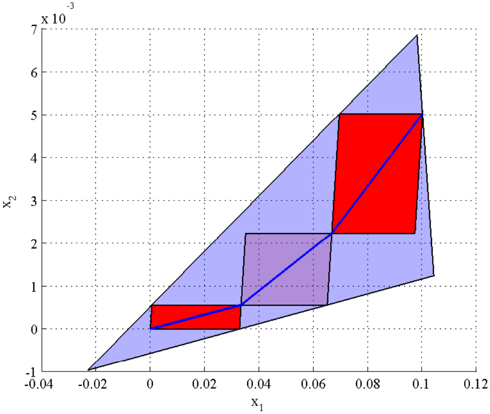


How the algorithm works

Which of them has the biggest volume?



How the algorithm works

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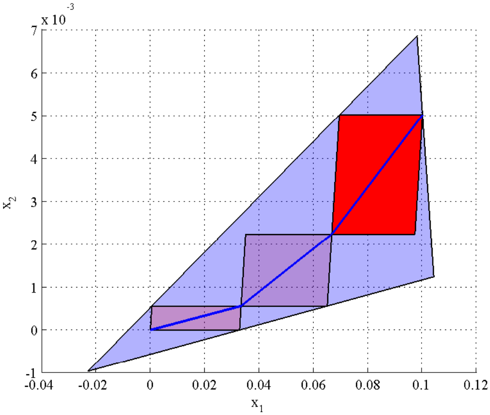
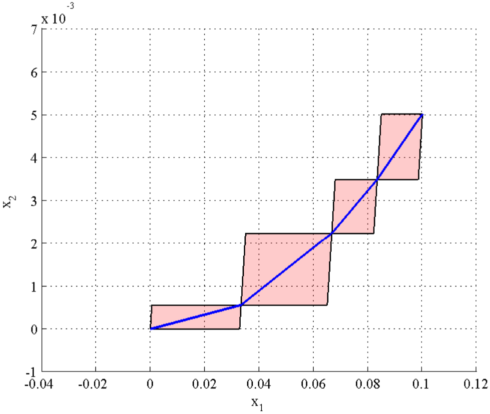


Re-sample the subinterval



How the algorithm works

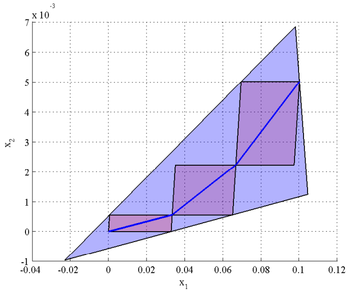


Figure: Volume: 0.3660×10^{-3}

How the algorithm works

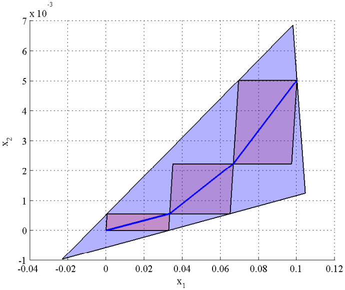


Figure: Volume: 0.3660×10^{-3}

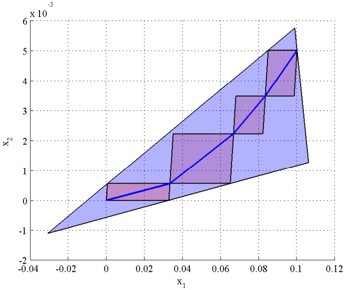


Figure: Volume: 0.3144×10^{-3}

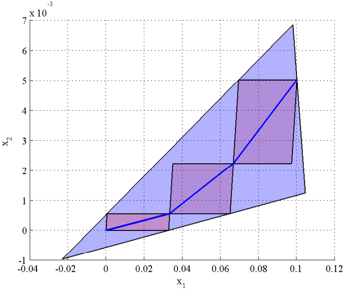


Figure: Volume: 0.3660×10^{-3}

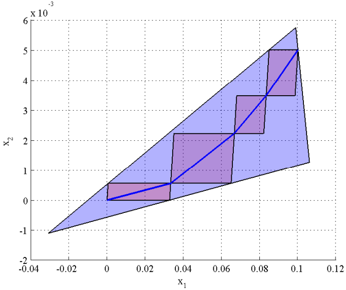


Figure: Volume: 0.3144×10^{-3}

If (volume_{actual} - $\text{volume}_{anterior}$ $<$ precision), the algorithm stops.

Example: The algorithm.

$$A_c = \begin{bmatrix} -0.1 & 0.21 \\ -0.21 & -0.1 \end{bmatrix}; \quad B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\epsilon_{max} = 100s$$

$$\lambda_{1,2} = \begin{bmatrix} -0.1 + 0.21\ i \\ -0.1 - 0.21\ i \end{bmatrix}$$

Outline

Positive Invariant Sets

Definition

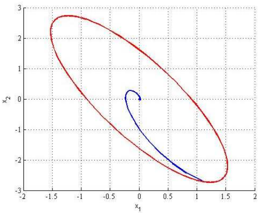
Blanchini (1999): A set P is said positively invariant for a system if for all $x(0) \in P$ the solution $x(t) \in P$ for $t > 0$. If $x(0) \in P$ implies $x(t) \in P$ for all $t \in \mathbb{R}$ then we say that P is invariant.

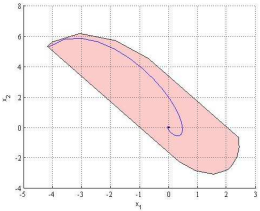
- Ellipsoidal invariant sets.

$$P = \{x \in \mathbb{R}^n \mid x^T P x < 1\}$$
$$P = P^T > 0$$

- Polyhedral invariant sets.

$$P = \{x \mid Cx \leq W\}$$





Robust Positive Invariant Set

Extended model:

$$\xi_{k+1} = A_\Delta \xi_k + B_\Delta u_k$$

with

$$\xi_k^T = \begin{bmatrix} x_k \\ u_{k-d} \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix}; \; A_\Delta = \begin{bmatrix} A & B-\Delta & \Delta & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}; \; B_\Delta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}$$

- Polytopic model with $n+1$ extreme realizations $\Delta \in Co\{\Delta_0, \dots, \Delta_n\}$
- Global polytopic model in an extended state space:

$$\begin{aligned} \xi_{k+1} &= A_\Delta \xi_k + B_\Delta u_k \\ A_\Delta &\in \Omega \\ \Omega &= \mathrm{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \dots A_{\Delta_n}\} \end{aligned}$$

Positive Invariant Sets

- The extension of maximal admissible sets (Gilbert et al. 1991) for polytopic models.
- Rewrite the constraints in terms of the augmented state ξ.

$$\Gamma\xi_k + Du_k \leq W$$

- Polyhedral domain, using the stabilizing control law $u_k = K\xi_k$:

$$P = \left\{\xi \in \mathbb{R}^{(n+d\cdot m)} \mid \ (\Gamma + DK)\xi \leq W\right\}$$

- Iterative construction of invariant sets:

$$\Omega_k = \{\xi(\Gamma + DK)\Phi^k \leq W, \ \forall 0 \leq i \leq k\}$$

$$\Omega_\infty = \{\xi(\Gamma + DK)\Phi^k \leq W, \ \forall i \geq 0\}$$

Conditions for finite determination

Theorem

Suppose the following assumptions:

1. *There is a common Lyapunov function that assures the asymptotic stability of the systems* $\xi_{k+1} = \Phi\xi_k, \Phi \in \Omega_K$
2. *The polytope* P *is bounded*
3. $0 \in int\ P$

Then O_∞^Ω *is finitely determined*

Stabilizing Control Law: Unconstrained Case

- $Q > 0$ and $R > 0 \rightarrow$ suitable weighting matrices.
- Stabilizing control law $u_k = K\xi_k \rightarrow$ classical LMI problem (Boyd et al.):

$$\min_{\gamma, S, Y} \gamma$$

$$\begin{bmatrix} S & SA_{\Delta_i}^T + Y^T B_\Delta^T & SQ^{1/2} & Y^T R^{1/2} \\ A_{\Delta_i} S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2} S & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \succeq 0, \quad S \succeq 0$$

for all $\quad i = 0, \ldots, n$ with $K = YS^{-1}$.

Outline

Lyapunov-Krasovskii candidate

Question: How to avoid the extended space representation?

Lyapunov-Krasovskii candidate

Question: How to avoid the extended space representation? One of the existing solution is the use of the Lyapunov-Krasovskii candidates to stabilize the system. The objective is to design a control law:

$$\mathbf{u}_k = K\mathbf{x}_k$$

Delay independent approach - work in progress

For the system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \Delta K\mathbf{x}_{k-d+1} + (B-\Delta)K\mathbf{x}_{k-d}$$

Lyapunov candidate:

$$V_k = \mathbf{x}_k^T P_0 \mathbf{x}_k + \sum_{i=1}^{d-1} \mathbf{x}_{k-i}^T P_1 \mathbf{x}_{k-i} + \mathbf{x}_{k-d}^T P_2 \mathbf{x}_{k-d} > 0$$

$$V_{k+1} - V_k \leq 0$$

Delay independent approach

Theorem

The state feedback matrix K is given by:

$$K = YG_0^{-1}$$

$$\min_{\gamma,\ G_0,\ G_x,\ G_y,\ Y} \quad \gamma$$

subject to:

$$\begin{bmatrix} G_0 & 0 & 0 & G_x & 0 & G_0 A^T & G_0 Q^{\frac{1}{2}} & 0 & 0 \\ 0 & G_x & 0 & 0 & G_y & Y^T \Delta^T & 0 & Y^T R_1^{\frac{1}{2}} & 0 \\ 0 & 0 & G_y & 0 & 0 & Y^T (B-\Delta)^T & 0 & 0 & Y^T R_2^{\frac{1}{2}} \\ G_x & 0 & 0 & G_x & 0 & 0 & 0 & 0 & 0 \\ 0 & G_y & 0 & 0 & G_y & 0 & 0 & 0 & 0 \\ AG_0 & \Delta Y & (B-\Delta)Y & 0 & 0 & G_0 & 0 & 0 & 0 \\ Q^{\frac{1}{2}} G_0 & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 & 0 \\ 0 & R_1^{\frac{1}{2}} Y & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 \\ 0 & 0 & R_2^{\frac{1}{2}} Y & 0 & 0 & 0 & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

Delay dependent approach - work in progress

For the system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_{k-d},$$

We have the Lyapunov-Krasovskii candidate:

$$V_k = V_{1k} + V_{2k} + V_{3k} \geq 0,$$

where V_{1k}, V_{2k} and V_{3k} are, respectively:

$$V_{1k} = \mathbf{x}_k^T P \mathbf{x}_k$$

$$V_{2k} = \sum_{i=-d}^{-1} \sum_{j=i}^{-1} (\mathbf{x}_{k+j+1}^T - \mathbf{x}_{k+j}^T) Z (\mathbf{x}_{k+j+1} - \mathbf{x}_{k+j})$$

$$V_{3k} = \sum_{i=-d}^{-1} (\mathbf{x}_{k+i}^T Q \mathbf{x}_{k+i}).$$

Delay dependent approach - work in progress

Theorem

Consider the linear discrete-time system with an uncertain input time-delay $d \in [0, d_{max}]$. If there exists the matrices $G > 0$, $J > 0$, H, T, W and L such that the following inequalities hold:

$$\begin{bmatrix} T + T^T + dH + J - G & -T & GA^T & dG(A-I)^T \\ -T^T & -J & L^T B^T & dL^T B^T \\ AG & BL & -G & 0 \\ d(A-I)G & dLB & 0 & -dW \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} H & T \\ T^T & GW^{-1}G \end{bmatrix} \geq 0,$$

then the system is asymptotically stable and the state feedback matrix K that stabilize the system is given by:

$$K = LG^{-1}.$$

Consider the autonomous system:

$$\xi_{k+1} = \tilde{A}_i \xi \text{ for } i = 1, \ldots, d$$

The term BK moves toward the matrix $\tilde{A}_i$ for $i = 1, \ldots, d$ in order to cover all the realizations of d.

$$\xi_k^T = \begin{bmatrix} x_k \\ u_{k-d} \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix} \quad ; \quad \tilde{A}_i = \left[\begin{array}{c|cccc} A & 0_{n\times(i-1)n} & BK & 0_{n\times(d-i)n} & \\ \hline I & 0 & \cdots & 0 & \\ 0 & I & \cdots & 0 & \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & 0 & I & 0 \end{array}\right]$$

Constraints Admissible Set - work in progress

Polyhedral set of constraints on the states and control:

$$P_x = \left\{ x \in \mathbb{R}^n \mid \underbrace{C + DK}_{H} x \leq \underbrace{W}_{h} \right\}.$$

Cross product between the regions P_x in an extended space:

$$P = \underbrace{P_x \times P_x \times \cdots \times P_x}_{d+1} = \left\{ \xi \in \mathbb{R}^{(d+1)n} \mid H\xi \leq h \right\}$$

Constraints Admissible Set - work in progress

For each dynamics $\tilde{A}_i, \forall i = 1, \ldots, d$ the maximal constraints admissible set is:

$$\Omega_i = \left\{ \xi_0 \mid \bigcup_{j=0}^{k} \tilde{A}_i^j \xi_0 \in P, \forall k \in \mathbb{N} \right\}$$

So a delay independent constraints admissible set Ω is:

$$\Omega = \bigcap_{i=1}^{d} \Omega_i$$

with $\Omega \subset P \subset \mathbb{R}^{(d+1)n}$.

Outline

Predictive Control

Standard MPC (Model Predictive Control):

- Predict the future system response.
- Optimal control policy $\rightarrow$ receding control horizon.
- Time-domain formulation $\rightarrow$ handling constraints and uncertainties.
- Prediction horizon larger than the delay

$$\mathbf{u}_k = \min_{u_k, u_{k+1}, \ldots, u_{k+N}} \mathbf{x}_{k+N}^T P \mathbf{x}_{k+N} + \sum_{i=0}^{N-1} \{\mathbf{x}_{k+i}^T Q \mathbf{x}_{k+i} + \mathbf{u}_{k+i}^T R \mathbf{u}_{k+i}\}$$

$$\text{subject to : } \; C\mathbf{x}_k + D\mathbf{u}_k \leq W$$

$$\mathbf{x}_{k+N} \in \Omega$$

Predictive Control

On-line optimization or analytic solution for parametric QP:

$$\mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} \mathbf{k}_u^T H \mathbf{k}_u + 2\mathbf{k}_u^T F \xi$$
$$\text{subject to :} \quad A_{in}\mathbf{k}_u \leq b_{in} + B_{in}\xi$$

Outline

Example

Example: Double Integrator

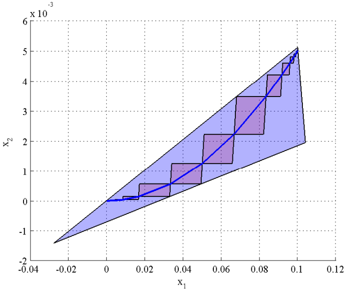
$$A_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T_e = 0.1s \quad \epsilon_{max} = 0.1s$$

$$\lambda_{1,2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

- By the algorithm presented:



- Precision: 1×10^{-7}
- Numerical results:

$$\Delta_0 = \begin{bmatrix} -0.0276 \\ -0.0014 \end{bmatrix} \quad \Delta_1 = \begin{bmatrix} 0.0999 \\ 0.0051 \end{bmatrix} \quad \Delta_2 = \begin{bmatrix} 0.1040 \\ 0.0019 \end{bmatrix}$$

Example

- Extended polytopic model:

$$\xi_{k+1} = A_{\Delta_i}\xi_k + B_{\Delta_i}u_k$$

with

$$A_{\Delta 1} = \begin{bmatrix} 1 & 0 & 0.1276 \\ 0.1 & 1 & 0.0064 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{\Delta 1} = \begin{bmatrix} -0.0276 \\ -0.0014 \\ 1 \end{bmatrix};$$

$$A_{\Delta 2} = \begin{bmatrix} 1 & 0 & 0.0001 \\ 0.1 & 1 & -0.0001 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{\Delta 2} = \begin{bmatrix} 0.0999 \\ 0.0051 \\ 1 \end{bmatrix};$$

$$A_{\Delta 3} = \begin{bmatrix} 1 & 0 & -0.0040 \\ 0.1 & 0.1 & 0.0031 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{\Delta 3} = \begin{bmatrix} 0.1040 \\ 0.0019 \\ 1 \end{bmatrix}$$

Example

- By solving the LMI:

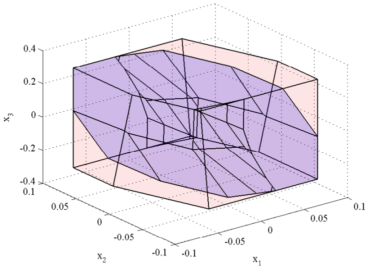
$$K = \begin{bmatrix} -3.8350 & -1.3096 & -0.0622 \end{bmatrix}$$

- Constraints in the input and states:

$$-0.3 \leq u_k \leq 0.3; \quad -0.08 \leq x_k \leq 0.08$$

- Positive invariant set:

$$H_0 \; x \leq K_0$$
$$t^* = 5$$



Example

For $Q = 0.1\ I_n$, $R = 0.00001$ and $N = 1$

Outline

Conclusions and Perspectives

Conclusions:

- New approaches to obtain polytopic models for time-delay systems
- Lyapunov-Krasovskii candidates (Just been accepted to ACC'10)
- Robust positive invariant sets in an extended state space (ECC'09)
- Constrained predictive control for time-varying delay systems (IFAC TDS'09)

Perspectives:

- Delay greater than sampling period (Work in progress)
- A new concept of invariant sets for time-delay system, inducing a Lyapunov-Razumikin function (CDC 2010), in cooperation with TUe/Eindhoven
- Application on the synchronization of mechanical systems in cooperation with Korea University