Polytopic methods, set invariance and predictive control for different classes of systems with variable time-delay

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Outline

Outline

Introduction

Time-delay systems (Niculescu 2001)**:**

- Complicated time behavior:
  - Oscillations
  - Instabilities
- Infinite dimension in continuous time

Present in:

- Propagation and transport phenomena
- Population dynamics (reproduction, development or extinction)
- Economic systems:
  - Investment policy
  - Market evolution
  - Analysis and decision
- Communication $\rightarrow$ *non-zero time interval* between the initiation and delivery of a signal.

Introduction

Objectives:

- Constrained control for LTI systems with variable and bounded time-delay

Tools:

- Time-delay system $\rightarrow$ Polytopic uncertain system
  - Embeddings $\rightarrow$ Convex bodies and convex geometry
  - Classical robust stabilization
- Lyapunov-Krasovskii candidate

Constrained control:

- Construct robust positive invariant sets
- Design constrained predictive control laws

Problem Formulation

- Continuous linear system with input delay:

$$\dot{x}(t) = A_c x(t) + B_c u(t-h)$$

- Degree of uncertainty: $h = dT_e - \epsilon$, with sampling time T_e.
- Discrete model:

$$x_{k+1} = Ax_k + Bu_{k-d} - \Delta(u_{k-d} - u_{k-d+1})$$

$$u(t) = u_k, \ \forall \ t \in [t_k, t_{k+1})$$

where:

$$A = e^{A_c T_e}, \quad B = \int_0^{T_e} e^{A_c(T_e-\theta)} B_c d\theta, \quad \Delta = \int_{-|\epsilon|}^{0} e^{-A_c\tau} B_c d\tau$$

- $\Delta \rightarrow$ exponential function in terms of the uncertainty ϵ.

Objectives

Objective:

- **Robust stability of LTI systems with time-variable delay**
- **Design a control law which regulates the system state while robustly satisfying a set of constraints:**

$$Cx_k + Du_k \leq W$$

Outline

Polytopic Embeddings

Goals:

- **Confine Δ in a polytope** $\rightarrow$ convex set covering all the possible realizations
- Basic element of construction: Jordan canonical form $A_c = V \Lambda V^{-1}$
- Low complexity polytopes $\rightarrow$ Simplex ($n + 1$ realizations)

Cases to be treated:

1. Non defective matrix A_c with real and non repeated eigenvalues
2. Non defective matrix A_c with complex conjugated eigenvalues
3. Defective matrix A_c with real and repeated eigenvalues

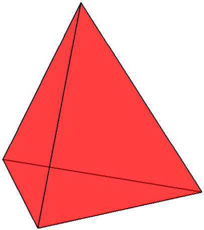
Simplex

Definition

Simplex or n-simplex: Is the convex hull of a set of $(n+1)$ affinely independent points in an Euclidean Space of n dimension.

Why the choice of a **simplex**?

- Reduced complexity in terms of extreme points
- "Volume" expressed by an analytical function



Non defective transfer matrix with real and non repeated eigenvalues

Example: Confining Δ.

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t-h)$$

$$\epsilon = 0.1$$

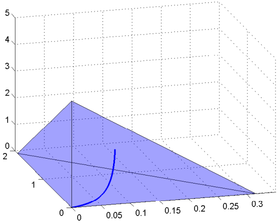
Extreme realizations:

$$\Delta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Delta_1 = \begin{bmatrix} 4.02 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 0 \\ 1.9 \\ 0 \end{bmatrix} \qquad \Delta_3 = \begin{bmatrix} 0.32 \\ 0 \\ 0 \end{bmatrix}$$

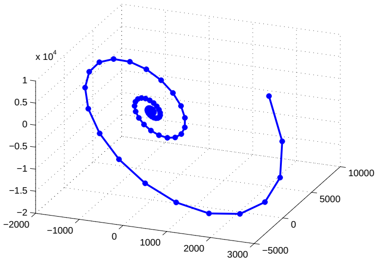
Using the result of Olaru et al. (2008):

$$\Delta_0 = \mathbf{0}_{n\times m};$$
$$\Delta_i = n\ V \int_0^{\bar{\epsilon}} e^{\Lambda_i \tau} d\tau V^{-1} B_c$$



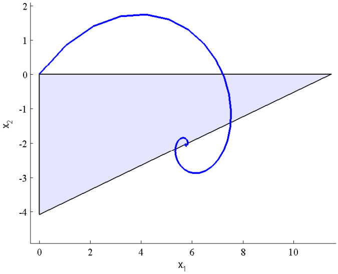
Non defective transfer matrix with complex conjugated eigenvalues

- Nonlinear dependence of $\Delta \rightarrow$ spiral type function
- Containment by a simplex is a hard task



Non defective transfer matrix with complex conjugated eigenvalues

- Nonlinear behavior of $\Delta \to$ spiral type
- Containment by a simplex is a hard task
- Using the existent techniques (Olaru et al. (2008)):



Non defective transfer matrix with complex conjugated eigenvalues

- By the Jordan decomposition:

$$\Delta = \int_{-|\epsilon|}^{0} e^{-A_c\tau} B_c d\tau = \underbrace{V\Lambda^{-1}e^{\Lambda\bar{\epsilon}}V^{-1}B_c}_{\theta} - V\Lambda^{-1}V^{-1}B_c$$

- **Extreme realizations**: Defining the center as $c_p = A_c^{-1}B_c$.
  1. A_c stable - Hypercube by forward dynamics $\dot{\theta} = \Lambda\theta$:

$$|\theta| \leq \left|V\Lambda^{-1}\right| \left|V^{-1}B_c\right|$$

  2. A_c instable - ϵ will "travel" along $[0, \bar{\epsilon}]$ from $\bar{\epsilon}$ to 0. Δ in the reverse time by $-A_c$ stable.

$$|\theta| \leq \left|V\Lambda^{-1}\right| e^{\Lambda\bar{\epsilon}} \left|V^{-1}B_c\right|$$

Hypercube constraints done by the box:

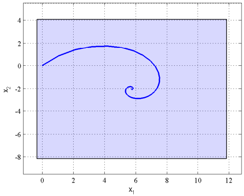
$$-|\theta| - c_p \leq \Delta_i \leq |\theta| + c_p$$

Non defective transfer matrix with complex conjugated eigenvalues

Example 1: A_c stable

$$\dot{x}(t) = \begin{bmatrix} -0.1 & 0.21 \\ -0.21 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-h)$$

$$\epsilon = 100; \quad \lambda_{1,2} = \begin{bmatrix} -0.1 + 0.21\ i \\ -0.1 - 0.21\ i \end{bmatrix}$$



Applying the results:

$$\begin{cases} c_p = A_c^{-1} B_c = \begin{bmatrix} 5.7301 \\ -2.0333 \end{bmatrix} \\ \left|V\Lambda^{-1}\right| \left|V^{-1} B_c\right| = \begin{bmatrix} 6.0802 \\ 6.0802 \end{bmatrix} \end{cases}$$

Box:

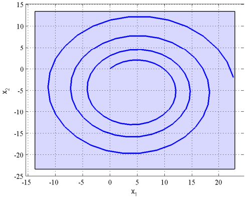
$$\begin{bmatrix} -0.3500 \\ -8.1135 \end{bmatrix} \leq |\theta| \leq \begin{bmatrix} 11.8103 \\ 4.0469 \end{bmatrix}$$

Non defective transfer matrix with complex conjugated eigenvalues

Example 2: A_c unstable

$$\dot{x}(t) = \begin{bmatrix} 0.1 & 0.21 \\ -0.21 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-h)$$

$$\epsilon = 100; \quad \lambda_{1,2} = \begin{bmatrix} 0.01 + 0.21\ i \\ 0.01 - 0.21\ i \end{bmatrix}$$



Applying the results:

$$\begin{cases} c_p = A_c^{-1} B_c = \begin{bmatrix} 4.5249 \\ -4.9774 \end{bmatrix} \\ \left| V \Lambda^{-1} \right| e^{\Lambda \bar{\epsilon}} \left| V^{-1} B_c \right| = \begin{bmatrix} 18.2851 \\ 18.2851 \end{bmatrix} \end{cases}$$

Box:

$$\begin{bmatrix} -13.7603 \\ -23.2625 \end{bmatrix} \leq |\theta| \leq \begin{bmatrix} 22.8100 \\ 13.3078 \end{bmatrix}$$

Defective transfer matrix with real eigenvalues

- Jordan decomposition $A_c = V\Sigma V^{-1}$ is the main ingredient.
- Existence of Jordan Blocks of multiplicity greater than 1.
- **Eigenvectors with linear dependence.**

Defective transfer matrix with real eigenvalues

- By the Jordan decomposition $A_c = V\Sigma V^{-1}$, and block diagonal matrix $\Sigma \in \mathbb{R}^{n\times n}$

$$\Sigma = \begin{bmatrix} \Sigma_{1,m_1} & 0 & \cdots & 0 \\ 0 & \Sigma_{2,m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{p,m_p} \end{bmatrix}; \quad \forall i \in [1, ..., p]$$

- Jordan blocks $\Sigma_{i,m_i} \in \mathbb{R}^{m_i\times m_i}$ with:

$$\Sigma_{i,m_i} = \begin{bmatrix} \sigma_i & 1 & \cdots & 0 & 0 \\ 0 & \sigma_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_i & 1 \\ 0 & 0 & \cdots & 0 & \sigma_i \end{bmatrix} = \sigma_i \underbrace{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}_{\Lambda_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Gamma_i}$$

Defective transfer matrix with real eigenvalues

- Explicit matrices:

$$\Sigma = \begin{bmatrix} \sigma_1\Lambda_1 + \Gamma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2\Lambda_2 + \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p\Lambda_p + \Gamma_p \end{bmatrix} =$$

$$= \sum_{i=1}^{p} \sigma_i \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Lambda_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{L_i} + \sum_{i=1}^{p} \underbrace{\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \Sigma_i & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}}_{G_i}$$

- Compact form:

$$\Sigma = \sum_{i=1}^{p} \sigma_i L_i + G_i$$

- Contain Δ done by:

$$\Delta(\epsilon) = \int_0^{\epsilon} e^{A_c\tau} B_c d\tau = \int_0^{\epsilon} e^{V\Sigma V^{-1}\tau} d\tau \; B_c = V \int_0^{\epsilon} e^{\Sigma\tau} d\tau \; V^{-1} B_c$$

Defective transfer matrix with real and repeated eigenvalues

- Take a closer look in the exponential term.
- Exploiting the structure of the Jordan blocks:

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} e^{\Gamma_i \tau}$$

- Taylor expantion of $e^{\Sigma_{i,m_i}\tau}$ up to the $(m_i - 1)^{th}$ term:

$$e^{\Sigma_{i,m_i}\tau} = e^{\sigma_i \Lambda_i \tau} \left(I + \frac{1}{1!}\Gamma_i \tau + \frac{1}{2!}\Gamma_i^2 \tau^2 + \ldots + \frac{1}{(m_i - 1)!}\Gamma_j^{m_i - 1} \tau^{m_i - 1} \right)$$

Defective transfer matrix with real and repeated eigenvalues

Extreme realizations: $\rightarrow$ Two cases:

1. Jordan blocks of size 1. Techniques of (Olaru et al. 2008):

$$\Delta_{j,1}(\epsilon) = V \int_0^{\epsilon} e^{\Sigma\tau} d\tau \ V^{-1} B_c = V L_j \int_0^{\epsilon} e^{\sigma_j \tau} d\tau \ V^{-1} B_c$$

2. Jordan blocks of size ≥ 2. Embedding:

$$\Delta_j(\epsilon) = V L_j \int_0^{\epsilon} e^{\sigma_j \tau} d\tau \ V^{-1} B_c$$

$$\Delta_{j,1}(\epsilon) = V \tfrac{1}{1!} G_j \int_0^{\epsilon} \tau e^{\sigma_j \tau} d\tau \ V^{-1} B_c$$

$$\vdots$$

$$\Delta_{j,m_j-1}(\epsilon) = V \tfrac{1}{(m_j-1)!} G_j^{m_j-1} \int_0^{\epsilon} \tau^{m_j-1} e^{\sigma_j \tau} d\tau \ V^{-1} B_c$$

Toward less conservative embeddings

- **Conservative approach** $\rightarrow$ reduce conservatism.
- Low complexity embeddings: **Hypercubes** $\rightarrow$ **Simplices**.
- Local embeddings can be constructed upon the splitting of Δ.

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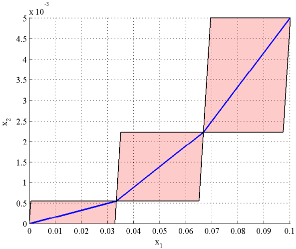


Toward less conservative embeddings

- Minimization of the *"volume"* of the simplex.

Theorem

The volume of a simplex S described by its bounding hyperplanes is given by:

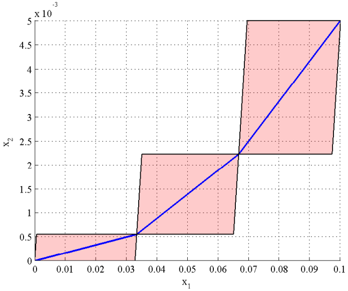
$$Vol(S) = \frac{|det(H)|^n}{n! \prod_{i=0}^{n} H_{i0}}$$

where H_{i0} is the cofactor of h_{i0} in H.

Proof: See (Gritzmann 1994) for details.

- Cost function taking into account the *"size"* of the convex hull.
- **Iterative algorithm to minimize the simplex volume at each step.**

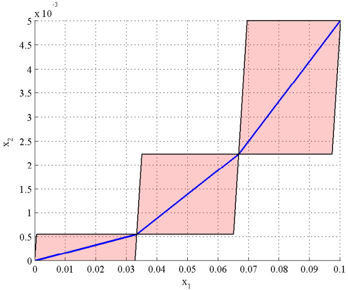
Result: Simplex volume minimized using the vertices of the local embeddings as constraints.

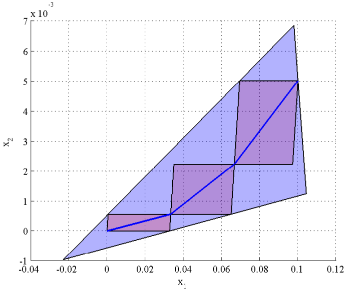


Minimizing the volume!

How the algorithm works

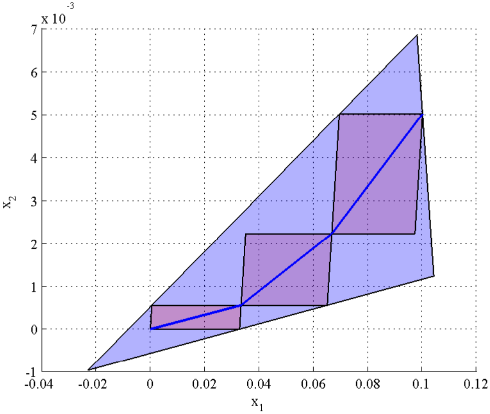
Result: Simplex volume minimized using the vertices of the local embeddings as constraints.



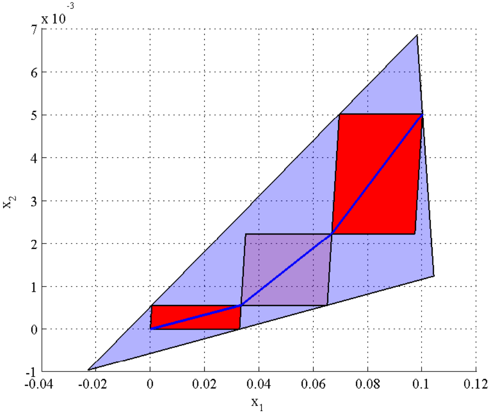


How the algorithm works

Which local embedding have the biggest number of vertices on the frontier of the polytopic embedding?

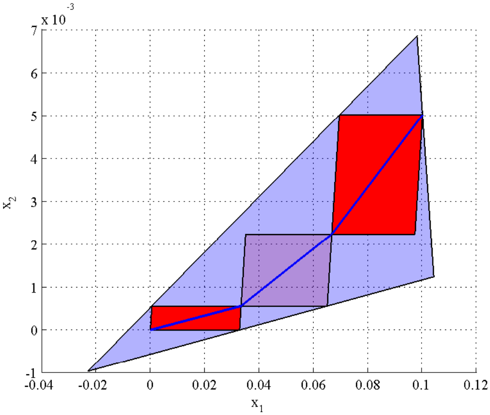


Which local embedding have the biggest number of vertices on the frontier of the polytopic embedding?



How the algorithm works

Which of them has the biggest volume?



How the algorithm works

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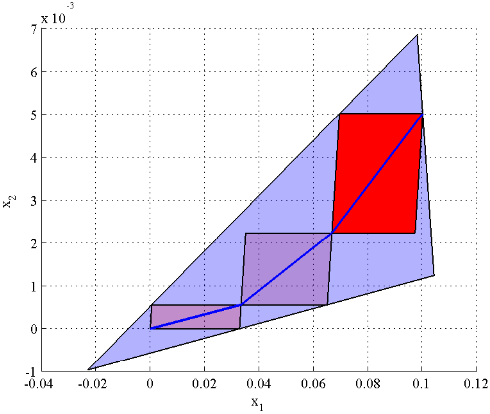
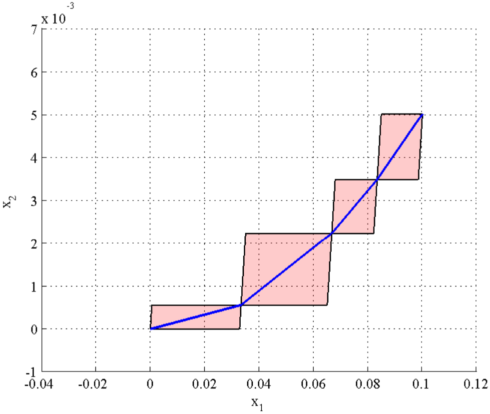


How the algorithm works

Re-sample the subinterval



How the algorithm works

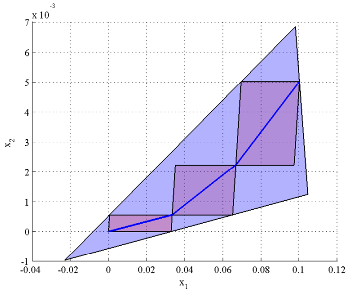


Figure: Volume: 0.3660×10^{-3}

How the algorithm works

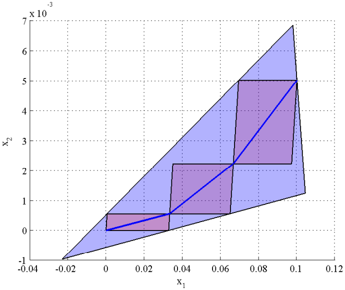


Figure: Volume: 0.3660×10^{-3}

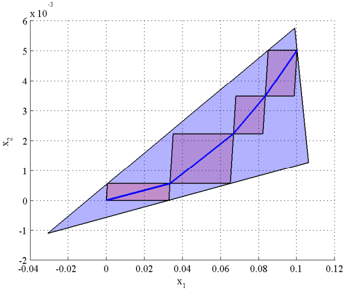


Figure: Volume: 0.3144×10^{-3}

How the algorithm works

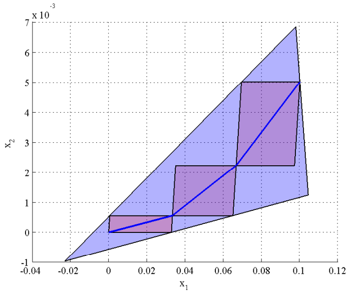


Figure: Volume: 0.3660×10^{-3}

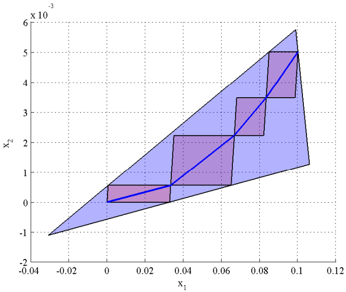


Figure: Volume: 0.3144×10^{-3}

If (volume$_{actual}$ - volume$_{anterior}$ $<$ precision), the algorithm stops.

Example: The algorithm.

$$A_c = \begin{bmatrix} -0.1 & 0.21 \\ -0.21 & -0.1 \end{bmatrix}; \quad B_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\epsilon_{max} = 100s$$

$$\lambda_{1,2} = \begin{bmatrix} -0.1 + 0.21\ i \\ -0.1 - 0.21\ i \end{bmatrix}$$

Outline

Positive Invariant Sets

Definition

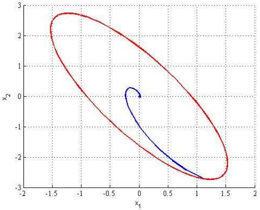
Blanchini (1999): A set P is said positively invariant for a system if for all $x(0) \in P$ the solution $x(t) \in P$ for $t > 0$. If $x(0) \in P$ implies $x(t) \in P$ for all $t \in \mathbb{R}$ then we say that P is invariant.

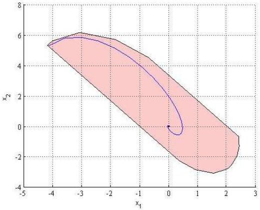
- Ellipsoidal invariant sets.

$$P = \{x \in \mathbb{R}^n \mid x^T P x < 1\}$$
$$P = P^T > 0$$

- Polyhedral invariant sets.

$$P = \{x \mid Cx \leq W\}$$





Robust Positive Invariant Set

Extended model:

$$\xi_{k+1} = A_\Delta \xi_k + B_\Delta u_k$$

with

$$\xi_k^T = \begin{bmatrix} x_k \\ u_{k-d} \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix}; \; A_\Delta = \begin{bmatrix} A & B-\Delta & \Delta & \dots & 0 \\ 0 & 0 & I_m & \dots & 0 \\ & & & \ddots & \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & 0 & \dots & I_m \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}; \; B_\Delta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}$$

- Polytopic model with $n+1$ extreme realizations $\Delta \in Co\{\Delta_0, \dots, \Delta_n\}$
- Global polytopic model in an extended state space:

$$\begin{aligned} \xi_{k+1} &= A_\Delta \xi_k + B_\Delta u_k \\ A_\Delta &\in \Omega \\ \Omega &= \mathrm{Co}\{A_{\Delta_0},\ A_{\Delta_1},\ \dots A_{\Delta_n}\} \end{aligned}$$

Positive Invariant Sets

- The extension of maximal admissible sets (Gilbert et al. 1991) for polytopic models.
- Rewrite the constraints in terms of the augmented state ξ.

$$\Gamma\xi_k + Du_k \leq W$$

- Polyhedral domain, using the stabilizing control law $u_k = K\xi_k$:

$$P = \left\{\xi \in \mathbb{R}^{(n+d\cdot m)} \mid \ (\Gamma + DK)\xi \leq W\right\}$$

- Iterative construction of invariant sets:

$$\Omega_k = \{\xi(\Gamma + DK)\Phi^k \leq W, \ \forall 0 \leq i \leq k\}$$

$$\Omega_\infty = \{\xi(\Gamma + DK)\Phi^k \leq W, \ \forall i \geq 0\}$$

Positive Invariant Sets

Conditions for finite determination

Theorem

Suppose the following assumptions:

1. *There is a common Lyapunov function that assures the asymptotic stability of the systems* $\xi_{k+1} = \Phi\xi_k, \Phi \in \Omega_K$
2. *The polytope* P *is bounded*
3. $0 \in int\ P$

Then O_∞^Ω *is finitely determined*

Stabilizing Control Law: Unconstrained Case

- $Q > 0$ and $R > 0 \rightarrow$ suitable weighting matrices.
- Stabilizing control law $u_k = K\xi_k \rightarrow$ classical LMI problem (Boyd et al.):

$$\min_{\gamma, S, Y} \gamma$$

$$\begin{bmatrix} S & SA_{\Delta_i}^T + Y^T B_\Delta^T & SQ^{1/2} & Y^T R^{1/2} \\ A_{\Delta_i} S + B_\Delta Y & S & 0 & 0 \\ Q^{1/2} S & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \succeq 0, \quad S \succeq 0$$

for all $\quad i = 0, \ldots, n$ with $K = YS^{-1}$.

Outline

Lyapunov-Krasovskii candidate

Question: How to avoid the extended space representation?

Lyapunov-Krasovskii candidate

Question: How to avoid the extended space representation? One of the existing solution is the use of the Lyapunov-Krasovskii candidates to stabilize the system. The objective is to design a control law:

$$\mathbf{u}_k = K\mathbf{x}_k$$

Delay independent approach - work in progress

For the system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + \Delta K\mathbf{x}_{k-d+1} + (B - \Delta)K\mathbf{x}_{k-d}$$

Lyapunov candidate:

$$V_k = \mathbf{x}_k^T P_0 \mathbf{x}_k + \sum_{i=1}^{d-1} \mathbf{x}_{k-i}^T P_1 \mathbf{x}_{k-i} + \mathbf{x}_{k-d}^T P_2 \mathbf{x}_{k-d} > 0$$

$$V_{k+1} - V_k \leq 0$$

Delay independent approach

Theorem

The state feedback matrix K is given by:

$$K = YG_0^{-1}$$

$$\min_{\gamma,\ G_0,\ G_x,\ G_y,\ Y} \quad \gamma$$

subject to:

$$\begin{bmatrix}
G_0 & 0 & 0 & G_x & 0 & G_0 A^T & G_0 Q^{\frac{1}{2}} & 0 & 0 \\
0 & G_x & 0 & 0 & G_y & Y^T \Delta^T & 0 & Y^T R_1^{\frac{1}{2}} & 0 \\
0 & 0 & G_y & 0 & 0 & Y^T (B-\Delta)^T & 0 & 0 & Y^T R_2^{\frac{1}{2}} \\
G_x & 0 & 0 & G_x & 0 & 0 & 0 & 0 & 0 \\
0 & G_y & 0 & 0 & G_y & 0 & 0 & 0 & 0 \\
AG_0 & \Delta Y & (B-\Delta)Y & 0 & 0 & G_0 & 0 & 0 & 0 \\
Q^{\frac{1}{2}} G_0 & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 & 0 \\
0 & R_1^{\frac{1}{2}} Y & 0 & 0 & 0 & 0 & 0 & \gamma I & 0 \\
0 & 0 & R_2^{\frac{1}{2}} Y & 0 & 0 & 0 & 0 & 0 & \gamma I
\end{bmatrix} \geq 0$$

Delay dependent approach - work in progress

For the system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_{k-d},$$

We have the Lyapunov-Krasovskii candidate:

$$V_k = V_{1k} + V_{2k} + V_{3k} \geq 0,$$

where V_{1k}, V_{2k} and V_{3k} are, respectively:

$$V_{1k} = \mathbf{x}_k^T P \mathbf{x}_k$$

$$V_{2k} = \sum_{i=-d}^{-1} \sum_{j=i}^{-1} (\mathbf{x}_{k+j+1}^T - \mathbf{x}_{k+j}^T) Z (\mathbf{x}_{k+j+1} - \mathbf{x}_{k+j})$$

$$V_{3k} = \sum_{i=-d}^{-1} (\mathbf{x}_{k+i}^T Q \mathbf{x}_{k+i}).$$

Delay dependent approach - work in progress

Theorem

Consider the linear discrete-time system with an uncertain input time-delay $d \in [0, d_{max}]$. If there exists the matrices $G > 0$, $J > 0$, H, T, W and L such that the following inequalities hold:

$$\begin{bmatrix} T + T^T + dH + J - G & -T & GA^T & dG(A-I)^T \\ -T^T & -J & L^T B^T & dL^T B^T \\ AG & BL & -G & 0 \\ d(A-I)G & dLB & 0 & -dW \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} H & T \\ T^T & GW^{-1}G \end{bmatrix} \geq 0,$$

then the system is asymptotically stable and the state feedback matrix K that stabilize the system is given by:

$$K = LG^{-1}.$$

Constraints Admissible Set - work in progress

Consider the autonomous system:

$$\xi_{k+1} = \tilde{A}_i \xi \text{ for } i = 1, \dots, d$$

The term BK moves toward the matrix $\tilde{A}_i$ for $i = 1, \dots, d$ in order to cover all the realizations of d.

$$\xi_k^T = \begin{bmatrix} x_k \\ u_{k-d} \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix} \; ; \quad \tilde{A}_i = \left[\begin{array}{c|cccc} A & 0_{n\times(i-1)n} & BK & 0_{n\times(d-i)n} & \\ \hline I & 0 & \cdots & 0 & \\ 0 & I & \cdots & 0 & \\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \cdots & 0 & I & 0 \end{array}\right]$$

Constraints Admissible Set - work in progress

Polyhedral set of constraints on the states and control:

$$P_x = \left\{ x \in \mathbb{R}^n \middle| \underbrace{C + DK}_{H} x \leq \underbrace{W}_{h} \right\}.$$

Cross product between the regions P_x in an extended space:

$$P = \underbrace{P_x \times P_x \times \cdots \times P_x}_{d+1} = \left\{ \xi \in \mathbb{R}^{(d+1)n} \middle| \; H\xi \leq h \right\}$$

Constraints Admissible Set - work in progress

For each dynamics $\tilde{A}_i, \forall i = 1, \dots, d$ the maximal constraints admissible set is:

$$\Omega_i = \left\{ \xi_0 \,|\, \bigcup_{j=0}^{k} \tilde{A}_i^j \xi_0 \in P, \forall k \in \mathbb{N} \right\}$$

So a delay independent constraints admissible set Ω is:

$$\Omega = \bigcap_{i=1}^{d} \Omega_i$$

with $\Omega \subset P \subset \mathbb{R}^{(d+1)n}$.

Outline

Predictive Control

Standard MPC (Model Predictive Control):

- Predict the future system response.
- Optimal control policy $\rightarrow$ receding control horizon.
- Time-domain formulation $\rightarrow$ handling constraints and uncertainties.
- Prediction horizon larger than the delay

$$\mathbf{u}_k = \min_{u_k, u_{k+1}, \ldots, u_{k+N}} \mathbf{x}_{k+N}^T P \mathbf{x}_{k+N} + \sum_{i=0}^{N-1} \{\mathbf{x}_{k+i}^T Q \mathbf{x}_{k+i} + \mathbf{u}_{k+i}^T R \mathbf{u}_{k+i}\}$$

$$\text{subject to : } \; C\mathbf{x}_k + D\mathbf{u}_k \leq W$$

$$\mathbf{x}_{k+N} \in \Omega$$

Predictive Control

On-line optimization or analytic solution for parametric QP:

$$\begin{aligned} \mathbf{k}_u^* &= \arg\min_{\mathbf{k}_u} \mathbf{k}_u^T H \mathbf{k}_u + 2\mathbf{k}_u^T F \xi \\ \text{subject to :} &\quad A_{in}\mathbf{k}_u \leq b_{in} + B_{in}\xi \end{aligned}$$

Outline

Example

Example: Double Integrator

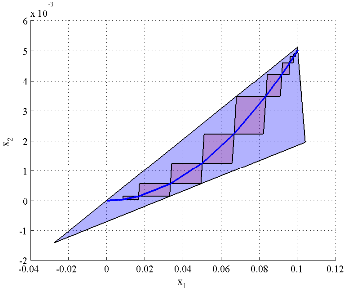
$$A_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T_e = 0.1s \quad \epsilon_{max} = 0.1s$$

$$\lambda_{1,2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

- By the algorithm presented:



- Precision: 1×10^{-7}
- Numerical results:

$$\Delta_0 = \begin{bmatrix} -0.0276 \\ -0.0014 \end{bmatrix} \quad \Delta_1 = \begin{bmatrix} 0.0999 \\ 0.0051 \end{bmatrix} \quad \Delta_2 = \begin{bmatrix} 0.1040 \\ 0.0019 \end{bmatrix}$$

Example

- Extended polytopic model:

$$\xi_{k+1} = A_{\Delta_i}\xi_k + B_{\Delta_i}u_k$$

with

$$A_{\Delta 1} = \begin{bmatrix} 1 & 0 & 0.1276 \\ 0.1 & 1 & 0.0064 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{\Delta 1} = \begin{bmatrix} -0.0276 \\ -0.0014 \\ 1 \end{bmatrix};$$

$$A_{\Delta 2} = \begin{bmatrix} 1 & 0 & 0.0001 \\ 0.1 & 1 & -0.0001 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{\Delta 2} = \begin{bmatrix} 0.0999 \\ 0.0051 \\ 1 \end{bmatrix};$$

$$A_{\Delta 3} = \begin{bmatrix} 1 & 0 & -0.0040 \\ 0.1 & 0.1 & 0.0031 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{\Delta 3} = \begin{bmatrix} 0.1040 \\ 0.0019 \\ 1 \end{bmatrix}$$

Example

- By solving the LMI:

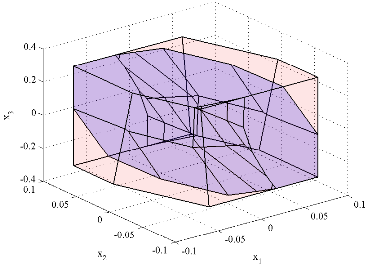
$$K = \begin{bmatrix} -3.8350 & -1.3096 & -0.0622 \end{bmatrix}$$

- Constraints in the input and states:

$$-0.3 \leq u_k \leq 0.3; \quad -0.08 \leq x_k \leq 0.08$$

- Positive invariant set:

$$H_0 \; x \leq K_0$$
$$t^* = 5$$



Example

For $Q = 0.1\ I_n$, $R = 0.00001$ and $N = 1$

Outline

Conclusions and Perspectives

Conclusions:

- New approaches to obtain polytopic models for time-delay systems
- Lyapunov-Krasovskii candidates (Just been accepted to ACC'10)
- Robust positive invariant sets in an extended state space (ECC'09)
- Constrained predictive control for time-varying delay systems (IFAC TDS'09)

Perspectives:

- Delay greater than sampling period (Work in progress)
- A new concept of invariant sets for time-delay system, inducing a Lyapunov-Razumikin function (CDC 2010), in cooperation with TUe/Eindhoven
- Application on the synchronization of mechanical systems in cooperation with Korea University