Systèmes à temps discret variant dans le temps avec vitesse bornée : analyse de stabilité et synthèse de commande

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This work has been developed with

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2. Most part of the presentation can be found in [OP09].

Abstract

This talk is concerned with

- Discrete-time linear systems with time-varying parameters;
- Parameters lie inside a polytope;
- Known bounds on their rate of variation.

Results — Robust Stability

- Convex model (polytope) to represent parameters and their variations;
- Linear matrix inequality conditions;
- Parameter-dependent Lyapunov function with polynomial dependence on the parameters.

Extensions

- Robust control design;
- Gain-scheduling by state feedback.

Motivation

Uncertain linear systems

- Robust stability analysis & control design;
- Parameter-dependent Lyapunov functions & other methods.

Time-varying case

- The use of parameter-dependent Lyapunov functions that consider finite bounds on the rates of the parameter variations provided less conservative results than the methods assuming arbitrary parameter variation (e.g. quadratic stability approach).
- The time-derivative of the parameter is modeled in a convex set and the bounds are included in the LMI conditions.

Discrete-time systems

Parameters can vary arbitrarily fast (including switching):

- Quadratic stability (analysis and control design);
- Affine parameter-dependent Lyapunov function [DB01];
- Path-dependent Lyapunov functions [LD06],[Lee06].

Parameters have bounds on their rate of variation:

- Naturally arises (and is related to the sampling time) when the discrete-time model is obtained by discretization;
- As an example, the quantity of fuel in an aircraft ranges from a minimum value to a maximum one (full tank) with rates of variation imposed by the physical limits of the turbines;
- A similar situation occurs when linear parameter varying models are obtained from a set of operation points by interpolation.

Discrete-time systems stability

- System: $x(k+1) = A(\alpha(k))x(k), \qquad A(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) A_i$

$$\alpha(k) = (\alpha_1(k), \alpha_2(k), \ldots, \alpha_N(k))' \in \Lambda_N, \qquad \forall 0 \leq k \in \mathbb{N}$$

$$\Lambda_N = \left\{ \delta \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0,\ i = 1, \ldots, N \right\}.$$

Theorem (Lyapunov)

The system is stable for all $\alpha(k) \in \Lambda_N$ *if* $\exists\, P(\alpha(k))$ *such that*

$$\begin{bmatrix} P(\alpha(k)) & A(\alpha(k))' P(\alpha(k+1)) \\ \star & P(\alpha(k+1)) \end{bmatrix} > \mathbf{0}$$

Remarks

- The proof follows from the first difference of the Lyapunov function $v(x(k),\alpha(k)) = x(k)'P(\alpha(k))x(k)$ computed along the trajectories of the system and Schur complement.

- In order to transform the parameter-dependent LMIs into LMIs one needs to impose a particular structure to the Lyapunov matrix $P(\alpha(k))$. For instance, $P(\alpha(k)) = P$ (quadratic stability) or affine in $\alpha(k)$, i.e. $P(\alpha(k)) = \alpha_1(k)P_1 + \cdots + \alpha_N(k)P_N$.

- Two different situations: (i) $\alpha(k)$ varies arbitrarily from k to $k+1$; (ii) $\alpha(k)$ has a known bound on the rate of variation.

Bounded rates of variation

In the discrete-time case, the maximum allowed variation depends on the actual value of $\alpha(k)$.

Parameter variation modeling

- Consider again the system

$$x(k+1) = A(\alpha(k))x(k) \qquad \alpha(k) \in \Lambda_N \ , \ \ A(\alpha) \in \mathscr{A}$$

The rate of variation of the uncertain parameters is given by

$$\Delta\alpha_i(k) = \alpha_i(k+1) - \alpha_i(k), \quad i = 1, \ldots, N$$

and, since $\alpha(k) \in \Delta_N$, one has

$$\sum_{i=1}^{N} \Delta\alpha_i(k) = 0.$$

Firstly, we assume that $\Delta\alpha_i(k)$ is bounded and satisfy the following condition

$$-b \leq \Delta\alpha_i(k) \leq b, \quad b \in \mathbb{R}, \ b \in [0,1].$$

Remarks

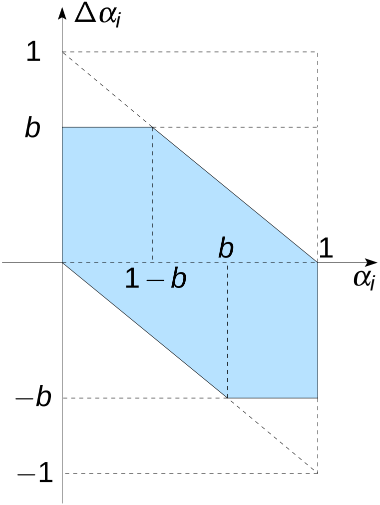
Bounded rates of variation

- The case $b = 0$ corresponds to uncertain but fixed parameters, i.e. the classic case of robust stability of uncertain discrete-time time-invariant systems in polytopic domains. Convergent LMI relaxations based on the existence of homogeneous polynomial solutions are available in the literature;
- The case $b = 1$, where the parameters are allowed to vary arbitrarily inside Λ_N from the instant k to the instant $k + 1$, has been handled by constant, affine and multi-affine (with guaranteed convergence) parameter-dependent Lyapunov functions in the literature;
- The main concern here is to investigate the case $0 < b < 1$ for polytopic systems.

Parameter variation modeling

- If one considers that the values of $\Delta\alpha_i(k)$ are independent of $\alpha_i(k)$, additional conservativeness is introduced.

As illustrated in the figure, the feasible values of $\Delta\alpha_i(k)$ (darken area) depend on the actual values of $\alpha_i(k)$.



Parameter variation modeling — Cartesian product

- For simplicity, the same b is considered for all $\Delta\alpha_i$, $i = 1, \ldots, N$.

- In the figure, any feasible pair $(\alpha_i, \Delta\alpha_i)$ belongs to the polytope Γ_i given by

$$\Gamma_i \triangleq \left\{ \delta \in \mathbb{R}^2 : \delta = \sum_{j=1}^{6} \lambda_j h_j, \quad \lambda \in \Lambda_6 \right\}$$

$$\begin{bmatrix} h_1 & \cdots & h_6 \end{bmatrix} = \left[\begin{array}{c:c:c:c:c:c} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{array}\right]$$

that is, Γ_i is the convex hull of the extremes (vertices) of the feasible area.

Parameter variation modeling — γ-space

- To model the $(\alpha_1, \ldots, \alpha_N, \Delta\alpha_1, \ldots, \Delta\alpha_N)$-space or, for simplicity, the $(\alpha, \Delta\alpha)$-space, one must take the Cartesian product of all Γ_i, $i = 1, \ldots, N$. The resulting vertices that do not obey

$$\sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \geq 0, \ i = 1, \ldots, N, \qquad \sum_{i=1}^{N} \Delta\alpha_i = 0$$

must be discarded. The resulting polytope, called Γ, is then given in the general case by

$$\Gamma = \left\{ \delta \in \mathbb{R}^{2N} : \delta = \sum_{i=1}^{M} \lambda_i h_i, \quad \lambda \in \Lambda_M \right\}$$

where $h_i \in \mathbb{R}^{2N}$ are given vectors. Generically, the number of vertices M grows as a function of N.

In this new modeling, both α and $\Delta\alpha$ are embedded together in an augmented space, called γ-space, of dimension $2N$.

Parameter variation modeling — example

- For instance, for $N = 2$, one has that the vertices of the polytope Γ constructed through $\Gamma_1 \times \Gamma_2$ (conveniently reordered) are given by

$$\begin{bmatrix} h_1 & h_2 & \dots & h_M \end{bmatrix} = \left[\begin{array}{cccc} f_1 & f_2 & \dots & f_M \\ \hline g_1 & g_2 & \dots & g_M \end{array}\right] = \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & b & 1-b \\ 0 & 0 & 1 & 1 & 1-b & b \\ \hline 0 & -b & 0 & b & -b & b \\ 0 & b & 0 & -b & b & -b \end{array}\right],$$

with $M = 6$.

- Thus, the first step to search for a solution to any robust LMI depending on both α and $\Delta\alpha$ is to lift the LMIs to the γ-space.

Parameter variation modeling — example

- Using the definition of Γ, it can be noted that $(\alpha, \Delta\alpha)$ and γ are related by a linear transformation.

$$\begin{pmatrix} \alpha \\ \Delta\alpha \end{pmatrix} = H\gamma, \quad H = \begin{bmatrix} h_1 & \cdots & h_M \end{bmatrix}, \quad \gamma \in \Lambda_M.$$

or, more precisely,

$$\alpha = \sum_{j=1}^{M} f_j \gamma_j, \qquad \Delta\alpha = \sum_{j=1}^{M} g_j \gamma_j$$

In the example, one has

$$\begin{aligned} \alpha_1 &= \gamma_1 + \gamma_2 + (1-b)\gamma_5 + b\gamma_6, & \alpha_2 &= \gamma_3 + \gamma_4 + b\gamma_5 + (1-b)\gamma_6 \\ \Delta\alpha_1 &= -b\gamma_1 + b\gamma_4 + b\gamma_5 - b\gamma_6, & \Delta\alpha_2 &= b\gamma_1 - b\gamma_4 - b\gamma_5 + b\gamma_6. \end{aligned}$$

Lyapunov functions for robust stability

Classes of Lyapunov parameter-dependent matrices

- Multi-affine on $\alpha(k)$ at successive instants of time from k until $k+L$ (*path-dependent*).
- Homogeneous polynomials of degree g.

$$P_g(\alpha) = \sum_{k \in \mathscr{K}_N(g)} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_N^{k_N} P_k$$

where $\mathscr{K}_N(g)$ be the set of N-tuples obtained from all possible combinations of N nonnegative integers with sum g.

$$P_L(\alpha(k)) = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_L=1}^{N} \alpha_{i_1}(k) \alpha_{i_2}(k+1) \times \cdots \alpha_{i_L}(k+L-1) P_{(i_1,i_2,\ldots,i_L)}$$

- Combinations of the two above (i.e. multipolynomials at successive instants of time).

Lifting to the γ-space

- Once a particular structure for the Lyapunov matrix is fixed, matrices $A(\alpha)$, $P(\alpha)$ and $P(\alpha + \Delta\alpha)$ must be written in terms of $\gamma \in \Lambda_M$.

For the path-dependent structure, $\Delta\alpha(k+1)$,$\Delta\alpha(k+2)$,$\ldots$, $\Delta\alpha(k+L-1)$ have to be expressed in terms of γ.

The resulting polynomial in the γ-space is also a homogeneous polynomial of degree g or L with M parameters. To search for a feasible solution, any LMI relaxations available in the literature to test the positivity of a homogeneous polynomial with variables in the simplex can be used.

Next theorem present LMI relaxations based on [OP07] to solve the robust LMIs associated to the Lyapunov stability conditions through the homogeneous polynomial Lyapunov matrix of arbitrary degree g.

LMIs for robust stability

Theorem

Assume that the vectors h_i, $i = 1, \ldots, M$ (related to Γ) are given. The system is robustly stable if there exists a degree $g > 0$, $g \in \mathbb{N}$, a relaxation level $d \in \mathbb{N}$ and symmetric matrices $P_k \in \mathbb{R}^{n \times n}$, $k \in \mathscr{K}_N(g)$ such that the following LMIs hold

$$L_k \triangleq \sum_{\substack{k' \in \mathscr{K}_M(d) \\ k \succeq k'}} \sum_{\substack{i \in \{1,\ldots,M\} \\ k_i > k'_i}} \frac{d!}{\pi(k')} \begin{bmatrix} \hat{P}_{k-k'-e_i} & \hat{A}'_i \tilde{P}_{k-k'-e_i} \\ \star & \tilde{P}_{k-k'-e_i} \end{bmatrix} > \mathbf{0},$$

$$\forall k \in \mathscr{K}_M(g+d+1)$$

where $\hat{A}_i$, $\hat{P}_k$ and $\tilde{P}_k$ are the coefficients of the homogeneous polynomial matrices $\hat{A}(\gamma)$, $\hat{P}_g(\gamma)$ and $\tilde{P}_g(\gamma)$ obtained with change of variables for $A(\alpha)$, $P_g(\alpha)$ and $P_g(\alpha + \Delta\alpha)$, respectively.

Robust State Feedback

Consider the discrete-time linear system

$$x(k+1) = A(\alpha(k))x(k) + B(\alpha(k))u(k)$$

where $\alpha(k)$ is supposed to be uncertain and $\Delta\alpha_i(k)$ is bounded. The system matrices are given in the form

$$\big(A(\alpha(k)), B(\alpha(k))\big) = \sum_{i=1}^{N} \alpha_i(k)(A_i, B_i), \quad \alpha(k) \in \Lambda_N.$$

The aim is to find a robust state feedback gain K such that the closed-loop system

$$x(k+1) = A_{cl}(\alpha(k))x(k),\ A_{cl}(\alpha(k)) = A(\alpha(k)) + B(\alpha(k))K$$

is robustly stable for all $(\alpha(k), \Delta\alpha(k)) \in \Gamma$.

LMIs for robust state feedback

Theorem

Assume that the vectors h_i, $i = 1, \ldots, M$ (related to Γ) are given. If there exists a degree $g > 0$, $g \in \mathbb{N}$, a relaxation level $d \in \mathbb{N}$, symmetric matrices $P_k \in \mathbb{R}^{n\times n}$, $k \in \mathscr{K}_N(g)$, matrices $G \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times n}$ such that the following LMIs hold

$$\sum_{\substack{k' \in \mathscr{K}_M(d) \\ k \succeq k'}} \frac{d!}{\pi(k')} \left(\begin{bmatrix} \tilde{P}_{k-k'} & \mathbf{0} \\ \star & -\hat{P}_{k-k'} \end{bmatrix} + \sum_{\substack{i \in \{1,\ldots,M\} \\ k_i > k'_i}} \frac{(g-1)!}{\pi(k-k'-e_i)} \begin{bmatrix} \mathbf{0} & \hat{A}_i G + \hat{B}_i Z \\ \star & G + G' \end{bmatrix} \right) > \mathbf{0}, \ \forall k \in \mathscr{K}_M(g+d)$$

where $\hat{A}_i$, $\hat{B}_i$, $\hat{P}_k$ and $\tilde{P}_k$ are the coefficients of the homogeneous polynomial matrices $\hat{A}(\gamma)$, $\hat{B}(\gamma)$, $\hat{P}_g(\gamma)$ and $\tilde{P}_g(\gamma)$, then $K = ZG^{-1}$ assures that the closed-loop system is robustly stable.

Gain-scheduled state feedback

- Suppose now that $\alpha(k)$ and $x(k)$ are available in real time for feedback by means of the state feedback control law

$$u(k) = K(\alpha(k))x(k), \quad K(\alpha(k)) \in \mathbb{R}^{m\times n} \text{ for all } k \geq 0.$$

The aim is to find a gain-scheduled state feedback gain $K(\alpha(k))$ such that the closed-loop system

$$x(k+1) = A_{cl}(\alpha(k))x(k), \quad A_{cl}(\alpha(k)) = A(\alpha(k)) + B(\alpha(k))K(\alpha(k))$$

is robustly stable for all $(\alpha(k), \Delta\alpha(k)) \in \Gamma$.

LMIs for gain-scheduled state feedback

Theorem

Assume h_i, $i = 1, \ldots, M$ are given. If there exists a degree $g > 0$, $g \in \mathbb{N}$, a relaxation level $d \in \mathbb{N}$, symmetric matrices $P_k \in \mathbb{R}^{n\times n}$, $k \in \mathscr{K}_N(g)$, matrices $G_k \in \mathbb{R}^{n\times n}$ and $Z_k \in \mathbb{R}^{m\times n}$, $k \in \mathscr{K}_N(g)$ such that

$$\sum_{\substack{k' \in \mathscr{K}_M(d) \\ k \succeq k'}} \sum_{\substack{i \in \{1,\ldots,M\} \\ k_i > k_i'}} \frac{d!}{\pi(k')} \begin{bmatrix} \tilde{P}_{k-k'-e_i} & \hat{A}_i \hat{G}_{k-k'-e_i} + \hat{B}_i \hat{Z}_{k-k'-e_i} \\ \star & \hat{G}_{k-k'-e_i} + \hat{G}'_{k-k'-e_i} - \hat{P}_{k-k'-e_i} \end{bmatrix} > \mathbf{0},$$

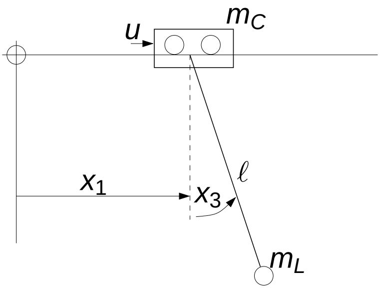
$$\forall k \in \mathscr{K}_M(g+d+1)$$

where $\hat{A}_i$, $\hat{B}_i$, $\hat{G}_k$, $\hat{Z}_k$, $\hat{P}_k$ and $\tilde{P}_k$ are the coefficients of $\hat{A}(\gamma)$, $\hat{B}(\gamma)$, $\hat{G}_g(\gamma)$, $\hat{Z}_g(\gamma)$, $\hat{P}_g(\gamma)$ and $\tilde{P}_g(\gamma)$, then $K(\alpha) = Z_g(\alpha) G_g(\alpha)^{-1}$ with

$$Z_g(\alpha) = \sum_{k \in \mathscr{K}_N(g)} \alpha_1^{k_1} \cdots \alpha_N^{k_N} Z_k, \; G_g(\alpha) = \sum_{k \in \mathscr{K}_N(g)} \alpha_1^{k_1} \cdots \alpha_N^{k_N} G_k$$

assures that the closed-loop system is robustly stable.

Example I — crane



The rope length ℓ belongs to an interval that represents the system operating from the point associated to the lifted container until the point where the load mass is placed down inside the ship.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_L g}{m_C} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-g(m_L+m_C)}{m_C \ell} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m_C} \\ 0 \\ \frac{-1}{m_C \ell} \end{bmatrix} u(t)$$

$$m_L = 1000 \text{ kg}, \; m_C = 300 \text{ kg}, \; g = 9.8 \; m/s^2$$

Example I — crane (cont.)

The aim is to design a gain-scheduled state-feedback stabilizing controller such that the uncertainty range of the rope is $\ell \in [2, \ell_{max}]$ with ℓ_{max} as large as possible.

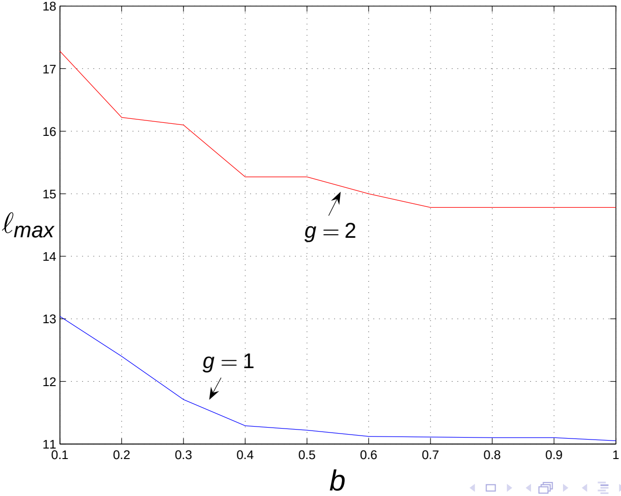
Additionaly, the state variable x_4 (angular velocity) is not available for feedback. Such requirement can be accomplished, for instance, by imposing the structural constraints on variables $Z_g(\alpha)$ and $G_g(\alpha)$ as follows

$$Z_g(\alpha) = \begin{bmatrix} z_1(\alpha) & z_2(\alpha) & z_3(\alpha) & 0 \end{bmatrix},$$

$$G_g(\alpha) = \begin{bmatrix} g_{11}(\alpha) & g_{12}(\alpha) & g_{13}(\alpha) & 0 \\ g_{21}(\alpha) & g_{22}(\alpha) & g_{23}(\alpha) & 0 \\ g_{31}(\alpha) & g_{32}(\alpha) & g_{33}(\alpha) & 0 \\ 0 & 0 & 0 & g_{44}(\alpha) \end{bmatrix}$$

Example I — crane (cont.)

A two vertices polytopic model was obtained by the discretization of system (Matlab routine c2d with $T_s = 0.2\ s$). The figure shows the values of ℓ_{max} obtained for $b \in [0.1, 1]$ and $g = 1, 2$.



Example II — control design

- Consider $n = 3$ and $N = 2$ and the two vertices system

$$\begin{bmatrix} A_1 & \vdots & A_2 \end{bmatrix} = \gamma \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 \end{array}\right], \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

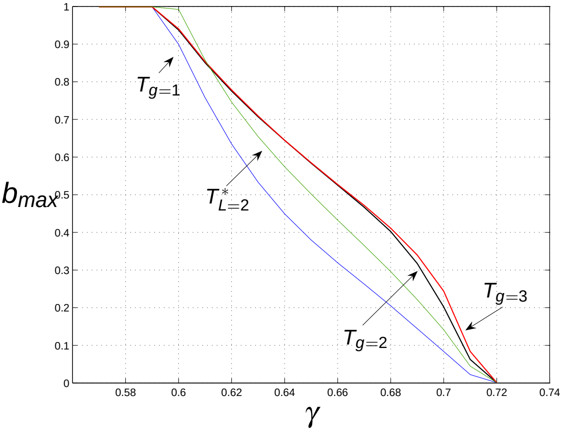
The aim is to determine the maximum value of γ such that the system is stabilizable by a robust state feedback gain.

For $b = 1$, i.e. the parameter α varies arbitrarily inside the polytope, both the conditions of our Theore and the conditions from [Mao03, Theorem 1] provide robust gains for $\gamma \leq 0.5940$.

For $\gamma > 0.5940$, the conditions from [Mao03, Theorem 1] can no longer provide feasible solutions.

Example II — control design (cont.)

The figure shows the maximum rates b_{max} for $\gamma \in [0.58\ 0.72]$ such that robust gains can be synthesized by our conditions with $g = 1, 2, 3$ and also the same conditions (denoted $T^*_{L=2}$) adapted to cope with the path-dependent structure with $L = 2$.



Conclusion

- The robust stability analysis and state feedback control design for time-varying discrete-time polytopic systems with bounded rates of parameter variation were presented.
- LMI conditions for robust stability analysis, robust and gain-scheduled state feedback control have been proposed for this class of systems, yielding less conservative results than other available techniques, as illustrated by examples.
- Performance requirements as the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms could be incorporated in the proposed parametrization without further ado.

Bibliography I

[DB01] J. Daafouz and J. Bernussou.
Parameter dependent Lyapunov functions for discrete time systems with time varying parameter uncertainties.
Systems & Control Letters, 43(5):355–359, August 2001.

[LD06] J.-W. Lee and G. E. Dullerud.
Uniform stabilization of discrete-time switched and Markovian jump linear systems.
Automatica, 42(2):205–218, February 2006.

[Lee06] J.-W. Lee.
On uniform stabilization of discrete-time linear parameter-varying control systems.
IEEE Transactions on Automatic Control, 51(10):1714–1721, October 2006.

Bibliography II

[Mao03] W. J. Mao.
Robust stabilization of uncertain time-varying discrete systems and comments on "an improved approach for constrained robust model predictive control".
Automatica, 39(6):1109–1112, June 2003.

[OP07] R. C. L. F. Oliveira and P. L. D. Peres.
Parameter-dependent LMIs in robust analysis: characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations.
IEEE Transactions on Automatic Control, 52(7):1334–1340, July 2007.

[OP09] R. C. L. F. Oliveira and P. L. D. Peres.
Time-varying discrete-time linear systems with bounded rates of variation: Stability analysis and control design.
Automatica, 45(11):2620–2626, November 2009.

Merci !