Pilotage des cycles limites dans les systèmes dynamiques hybrides. Application aux alimentations électriques statiques.

Diego A. Patino

Advisor: Claude Iung
Co-advisor: Pierre Riedinger

Centre de Recherche d'Automatique de Nancy
2009

Context

- Research Center in Automatic Control, Nancy
- Team ACOS (French: Automatique Commande et Observation de Systèmes)
  - Hybrid Dynamical Systems
- HYCON Project (Hybrid Control: Taming Heterogeneity and Complexity of Networked Embedded Systems.)
  - WP4a: Energy Management

$\Rightarrow$ Control of the cycles in SDH (switched system) with application in power converters.

Switched linear systems

Class of hybrid systems consisting in a family of linear subsystems:

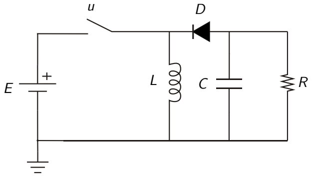
$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}E \qquad x(t) \in \mathbb{R}^n \Rightarrow \text{ State}$$

and a rule

$$\sigma : \mathbb{Z}^+ \rightarrow \{1, 2, \ldots, m\} \Rightarrow \text{mode}$$

that describes the sudden changes in the system behavior. (m : Number of modes)

Example: Power converters



$$x = [i_l, v_c]^T$$

$$\dot{x} = A_i x + B_i E, \qquad i = 1, 2$$

Open switch: $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

Closed switch: $A_2 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \qquad B_2 = \begin{bmatrix} \frac{E}{L} & 0 \end{bmatrix}^T$

$$\dot{x} = (1-u)(A_1 x + B_1 E) + u(A_2 x + B_2 E), \qquad u \in \{0, 1\}$$

Collecting terms with u

$$\Downarrow$$

$$\dot{x} = A_0(x) + B_0(x)u$$

In general, physical systems with ideal switches [Buisson et al., 2002]:

$$\dot{x} = f(x) + g(x)u, \qquad u \in U \subset \{0,1\}^r \tag{1}$$

$$x \in \mathbb{R}^n, \qquad f(x) \in \mathbb{R}^n, \qquad g(x) \in \mathbb{R}^{n\times r}$$

AFFINE IN THE CONTROL

$u \in co(U) \setminus U$

Example: Power converters

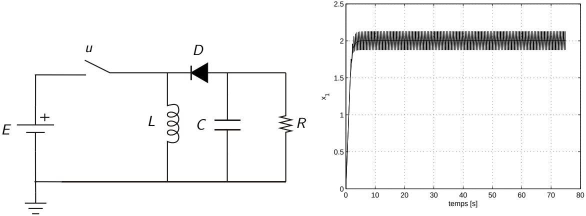


Figure: Buck-boost Converter

$$\dot{x} = A_0(x) + B_0(x)u$$

If $i_L = 2$ is desired in steady state, $u = 0.5$ $(R = L = C = E = 1)$

$$\Downarrow$$

It is not possible, $u \in \{0, 1\}$

Limit cycle

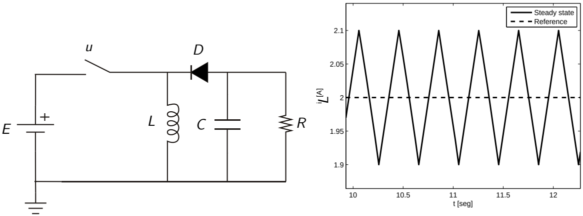


Figure: Buck-boost converter

$$\dot{x} = A_0(x) + B_0(x)u$$

If $i_L = 2$ is desired in steady state, $u = 0.5$ $(R = L = C = E = 1)$

$$\Downarrow$$

It is not possible, $u \in \{0, 1\}$

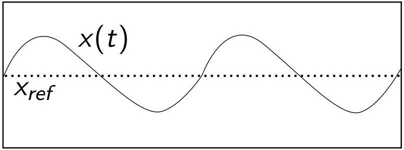
Cyclic switched systems and operation points

Definition

A cyclic switched system is a switched system that converges to a cycle.

Which are the performance specifications?
In steady state:

- Reach an average value.
- Follow an optimal cycle.



Steady state → cycle around an average value:

$$\overline{x} = \frac{1}{T_p} \int_{T_p} x(\tau) d\tau \qquad (2)$$

T_p: Period of the cycle.

Cyclic switched systems and operation points

To obtain the cycle average and describe a global dynamic behavior, the sliding average is used:

$$\overline{x}(t) = \frac{1}{T_p} \int_{t-T_p}^{t} x(\tau) d\tau \tag{3}$$

$\overline{u}(t)$: Sliding average value of u over a period. But:

- [Sanders and Verhulst, 1985]

$$\widetilde{x}(t) \Rightarrow \text{Average state model solution}$$
$$\widetilde{x}(t) \approx \overline{x}(t) \approx x(t), \text{ for } T_p \rightarrow 0$$

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))\overline{u}(t), \ \overline{u} \in co(U) \setminus U \tag{4}$$

General strategy

Operation points:

$$X_{ref} = \{x_{ref} \in \mathbb{R}^n : f(x_{ref}) + g(x_{ref})u_{ref} = 0, u_{ref} \in co(U) \setminus U\}$$

General strategy:

- Find a control law (switching law) that takes the system to the operation point: Transitory state. (**Optimal** or predictive control)
- Maintain the system in a neighborhood of the operation point: Cycle (Predictive control)

Constraint:

- Apply the control methods in real-time to power converters.

Problem formulation

Control problem:

$$\begin{aligned} &\min_{u(\cdot)} x_{n+1}(T) = \int_0^T L(x_1, x_2, \ldots, x_n) dt \\ s.t. \quad &\dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ &x(0) = x_0, x(T) \text{ free} \\ &u(t) \in U = \{0, 1\}^r \end{aligned}$$

Find the switching law $u^*(t) \in U$ for $t \in [0, T]$ which minimizes $x_{n+1}(T)$ (cost function) for any value of x_0.

All the switches are controlled, there is no time or state constrain on the control

IDEA: USE THE PONTRYAGIN MINIMUM PRINCIPLE [Pontryagin et al., 1964].

Necessary conditions

Define the Hamiltonian : $H(u,x,\lambda) = \lambda^T f(x) + \lambda^T g(x)u$

Theorem (Minimum principle)

If (x^*,λ^*,u^*) is optimal with $\dot{\lambda} = -H_x(u,x,\lambda)$, $\lambda \neq 0$ then for almost all $t \in [0,T]$

1. $H^* = \min_{u\in U} H(u,x^*,\lambda^*)$
2. $H(t) = cte (= 0 \text{ if } T \text{ is free})$
3. $\lambda^*(0)$ free and $\lambda^*(T) = [0,\ldots,0,1]^T$

- Hybrid versions of minimum Principle: [Sussman, 1999], [Dmitruk and Kaganovich, 2008], [Shaikh and Caines, 2003], [Riedinger and Iung, 1999], [Riedinger et al., 2003].
- For sake of simplicity ⇒ assumption: $r = 1$.

What can go wrong ?

- $H(u, x, \lambda) = \lambda^T f(x) + \lambda^T g(x)u$.
- Define the switching function $\phi(t) = \lambda^T g(x)$
- From the condition $\min_{u \in U} H(u, x^*, \lambda^*)$, u is given by

$$u = \begin{cases} 0 \text{ if } \phi(t) > 0 & \text{Bang - Bang control} \\ 1 \text{ if } \phi(t) < 0 & \text{Bang - Bang control} \\ ? \text{ if } \phi(t) = 0 & \forall t \in]a, b[\text{ Singular control} \end{cases}$$

- Why the singular solution is important?
  The operation points are inside the convex hull.

$$\{x_{ref} \in \mathbb{R}^n : f(x_{ref}) + g(x_{ref})u_{ref} = 0, u_{ref} \in co(U) \setminus U\}$$

THIS IS PRECISELY THE SINGULAR CONTROL CASE

Conditions to determine singular arcs

PROBLEM:

Search $\lambda(t)$, $x(t)$ and $u(t)$ such that $\phi(t) \equiv 0$.

$\phi(t)$ can be time-derived until u appears explicitly.

The problem order is the smallest integer q such that :

1. u appears explicitely in $\phi^{(2q)}$ (derivative of H_u w.r.t. t):

$$\phi^{(2q)} = H_u^{(2q)}(u, x, \lambda) = A(x, \lambda) + uB(x, \lambda)$$

2. If it doesn't exist then $q := \infty$. [Volker, 1996],[Robbins, 1967]

Conditions to determine the singular arcs

$$\phi(x, \lambda) = 0, \dot{\phi}(x, \lambda) = 0, \dot{\phi}^{(2)}(x, \lambda) = 0, \dots, \phi^{(2q-1)}(x, \lambda) = 0 \quad (5)$$

$$\phi^{(2q)} = A(x, \lambda) + uB(x, \lambda) = 0 \quad (6)$$

How to determine singular arcs? (Algebraically determined)

(5) and (6) give the conditions on x, λ and u.

- The switching function for the affine case is given by : $\phi(\lambda, x) = \lambda^T g(x)$.
- If the problem order is q then $2q < n$ and

$$(5) : \lambda^T ad_f^k g(x) = 0 \ \ \forall \ k = 0, 1, \ldots, 2q-1$$

are linearly independent [Fraser-Andrew, 1989].

$$(6) : \lambda^T ad_f^{2q} g(x) + u\lambda^T [g, ad_f^{2q-1} g](x) = 0$$

where
$ad_f^k g(x) = [f, ad_f^{k-1} g](x)$,$ad_f^0 g(x) = g(x)$: Iterate Lie brackets
$[f, g](x) := g_x(x) f(x) - f_x(x) g(x)$ Lie bracket

How to determine singular arcs? (Algebraically determined)

- If the problem order is q then

2q equations (5): $\lambda^T ad_f^k g(x) = 0 \;\; \forall \; k = 0, 1, \ldots, 2q-1$

+1 equation (6): $\lambda^T ad_f^{2q} g(x) + u\lambda^T [g, ad_f^{2q-1} g](x) = 0$

– – – – – ––

2q+1 equations

- From the minimum principle: $\lambda \neq 0$
- There are two cases where u can be written as a feedback $u = u(x)$ (Algebraically determined).

How to determine singular arcs? (Algebraically determined)

Case 1. If $n = 2q$: (5) and (6) yield:

1. $$u(x) = -\frac{det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), ad_f^{2q} g(x)])}{det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), [g, ad_f^{2q-1} g](x)])}.$$

2. and λ can be a-posteriori deduced from:

$$\lambda^T [g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x)] = 0. \qquad (7)$$

Case 2. For free time optimal control problems: $H = 0 \Rightarrow \lambda^T f(x) = 0$ (Optimal time and infinite quadratic criterion) gives a supplementary condition, then the control can be obtained for $n = 2q + 1$.

How to determine singular arcs?

In other cases, from $\lambda^T ad_f^{2q} g(x) + u\lambda^T [g, ad_f^{2q-1} g](x) = 0$, then $u = u(x, \lambda) \Rightarrow$ Two-boundary problem (Non-linear differential equation)

Optimal trajectories synthesis

1. Determine the singular arcs.
2. Keep only the singular arcs corresponding to $u(x) \in co(U)$ (since they can be "approximated" by $\{0, 1\}$).
3. Keep only singular arcs corresponding to a minimum of H.

Theorem

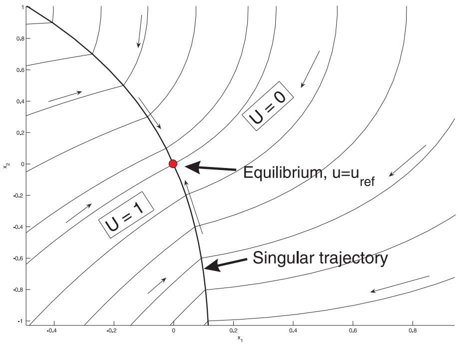
(2nd order condition - Legendre-Clebsch Condition) *Assume $q < \infty$ (most of cases) and let (x^*, u^*, λ^*) an optimal solution:*

- $(-1)^q B(x^*, \lambda^*) \geq 0$ *for all* $t \in]a, b[$.

4. Determine the regular arcs finishing in the singular optimal trajectory.

Optimal trajectories synthesis

Singular arc ⇒ Regular trajectories ⇒ Interpolation ⇒ Feedback control.



- The singular arc is composed by points (u_f, x_f, λ_f).
- From (u_f, x_f, λ_f), the Hamiltonian system:

$$\dot{x} = H_\lambda(u, x, \lambda)$$
$$\dot{\lambda} = -H_x(u, x, \lambda)$$

is backward integrated with $u = 0$ and $u = 1$.

Optimal trajectories synthesis

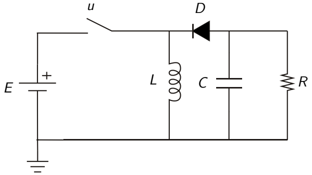
Singular arc ⇒ Regular trajectories ⇒ Interpolation ⇒ Feedback control.

The trajectories are interpolated using a neural network (Offline)

⇓

State feedback control $u(x)$

Example: Buck-boost converter



In a mayer form using a quadratic criterion,

1. Dimension $n = 2$
2. Problem order $q = 1$

$$f(x) = \begin{bmatrix} A_2 x + B_2 E \\ (x - x^{ref})^T Q (x - x^{ref}) \end{bmatrix}, g(x) = \begin{bmatrix} (A_1 - A_2)x + B_1 E - B_2 E \\ 0 \end{bmatrix}$$

1. The singular control is defined by

$$u(x) = -\frac{det([g(x), ad_f^1 g(x), ad_f^2 g(x)])}{det([g(x), ad_f^1 g(x), [g, ad_f^2 g](x)])},$$

$$\lambda^T \perp \{f(x), g(x), ad_f^1 g(x)\}$$

2. Legendre - Clebsch 2d order conditions :

$$-\lambda^T [g, ad_f g](x) \geq 0$$

Response of the Buck-boost converter + control

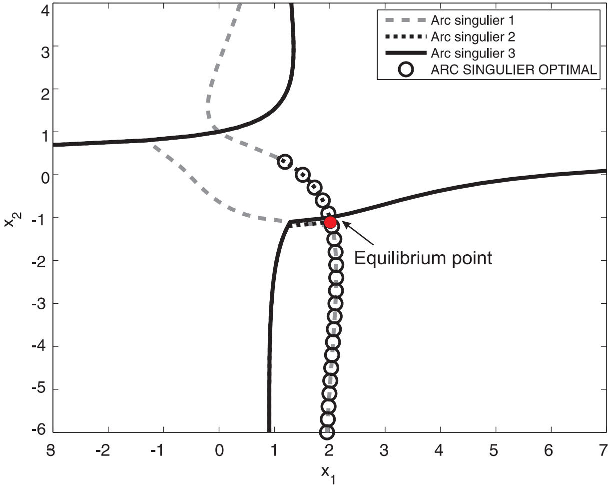
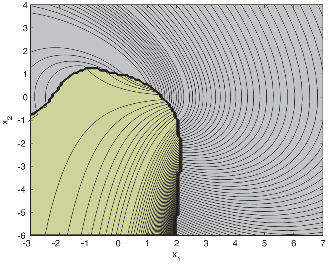
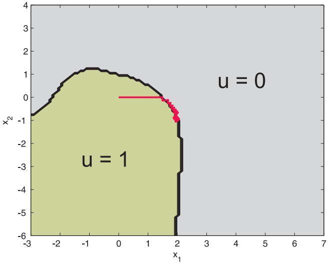
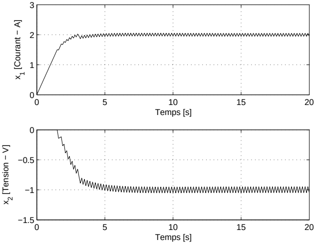


Figure: Verification of second order condition

Response of the Buck-boost converter + control







- Reference $x_{ref} = (2, -1)$
- Weight matrix

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$$

Remarks

- Multi-input case has been also developed.
- Average value in steady state is x_{ref}. The resulting cycle is a consequence and it is not controlled.
- Stability assured because of the construction.
- Takes some time to compute offline (10.45 seconds for the buck-boost, $1GB$ RAM.)

(CIFA 2008, IJC 2009)

Return to the general strategy

Operation points:

$$X_{ref} = \{x_{ref} \in \mathbb{R}^n : f(x_{ref}) + g(x_{ref})u_{ref} = 0, u_{ref} \in co(U) \setminus U\}$$

General strategy:

- Find a control law (switching law) that takes the system to the operation point: Transitory state. (Optimal or **predictive** control)
- Maintain the system in a neighborhood of the operation point: Cycle. (**Predictive** control)

Constraint:

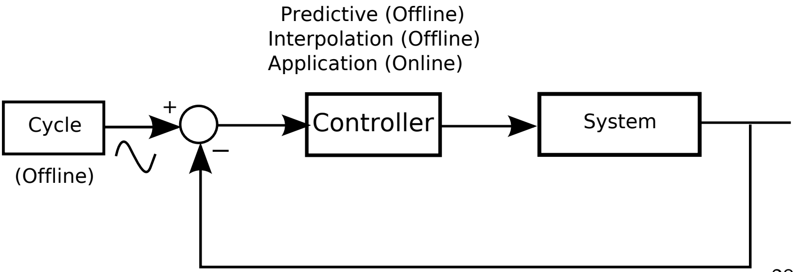
- Apply the control methods in real-time to power converters.

General idea

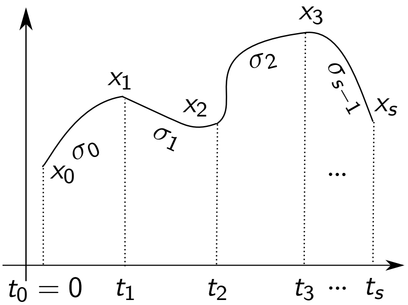
- The control system has usually a constant reference.
- Type of system: Cyclic systems

IDEA:

1. Steady state is a cycle ⇒ Reference can be also a cycle.
   Which cycle?
2. Reach this cycle in steady state. How?

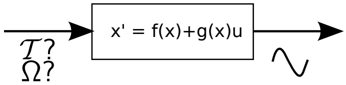


General idea to determine an optimal cycle



- Time sequence: $\mathcal{T} = \{t_0 = 0, t_1, t_2, \ldots, t_s\}$
- Mode sequence: $\Omega = \{\sigma_0, \sigma_1, \ldots, \sigma_{s-1}\}$
- State set: $\mathcal{X} = \{x_0, x_1, \ldots, x_s\}$

Open loop: Cycle research minimizing a cost function (Waveform).
We search $\mathcal{T}$ and Ω.



Optimal cycle is determined by non-linear programming.

Which cycle?: Cycle research

- Least oscillations criterion around a constant x_{ref} over a period T_p:

$$J(\mathcal{T}^{\infty}, \Omega^{\infty}, s^{\infty}) = \min_{\mathcal{T},\Omega,s} \int_0^{T_p} [x(t) - x_{ref}]^T \mathcal{Q}[x(t) - x_{ref}]dt \tag{8}$$

- Cycle constraint: $x(0) = x(T_p)$.
- Bound over the period: $T_p < T_{p,max}$
- Switching time constraint: The switch can change its state if $\delta_k \geq t_{min}$

$$\begin{aligned} \delta_k(t_i) \geq t_{min}|u_k(t_j) - u_k(t_{j-1})| & \qquad \forall j = 1, \dots, s \\ \dot{\delta}_k(t_j) = 1 & \qquad \forall k = 1, \dots, r \\ \delta_k(t_{j+1}) = 0 \qquad if & \qquad |u_k(t_j) - u_k(t_{j-1})| \neq 0 \end{aligned}$$

- How to reach the cycle?:
  1. Far from the cycle. The reference is the average value of the cycle.
  2. Near the optimal cycle. The reference is the cycle obtained before (Classical tracking problem).

Close loop construction

For each value of a initial condition set, take the modes and the time sequence which minimizes the cost function. (100×100)

$\Downarrow$

IT IS A VERY LONG PROCESS

(0.5 secs for each initial condition in a second-order system, a signal decision must be taken each $0.022ms$.)

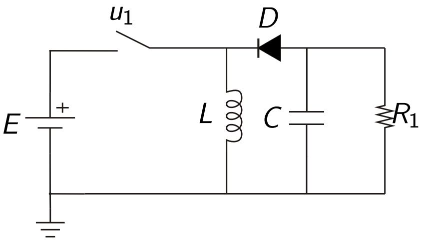
$\Downarrow$

INTERPOLATE THE SOLUTIONS FOR THE WHOLE SET OF INITIAL CONDITION.

$\Downarrow$

RESULT: A FEEDBACK CONTROL LAW.

Example: Buck-Boost converter

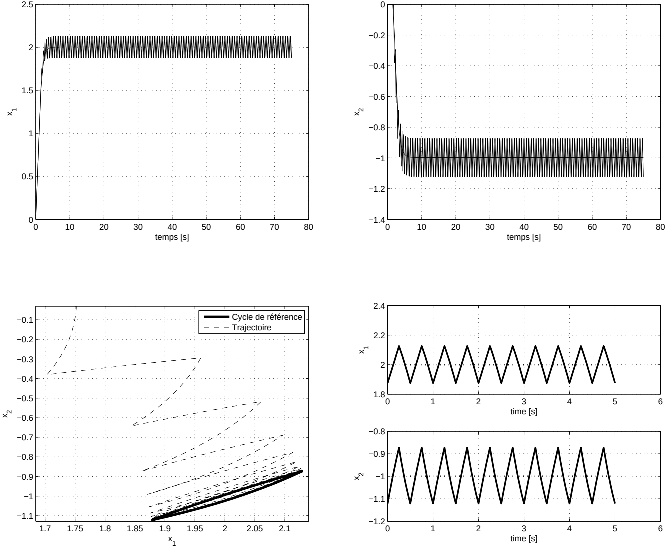


Parameters:

- $R = L = C = E = 1$, $\mathcal{Q} = diag[1, 1]$
- $f(x) = A_2 + B_2 E$ $g(x) = (A_1 - A_2)x + (B_1 - B_2)E$, $T_{p,max} = 1$ s
- $x_{ref} = [v_{c,ref}, i_{l,ref}]^T = [2, -1]^T$.
- $s_{max} = 3, \ t_{min} = 0.04s$

Optimal limit cycle: $\mathcal{T}^{\infty} = \{0, 0.2509, 0.5\}, \ \Omega^{\infty} = \{on, off\}$

Example: the Buck-Boost converter' response+control



Remarks

- Sensibility functions and a Gauss-Newton algorithm are used to find the switching times.
- Maintain the system in the neighborhood of the cycle... BUT nothing can be said about the stability!

(ACC 2007, JDMACS 2007, IECON 2008)

Multilevel converter

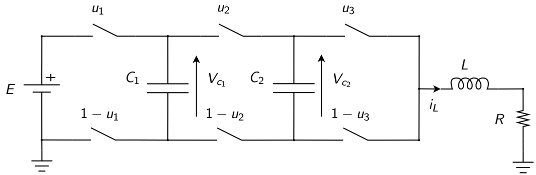
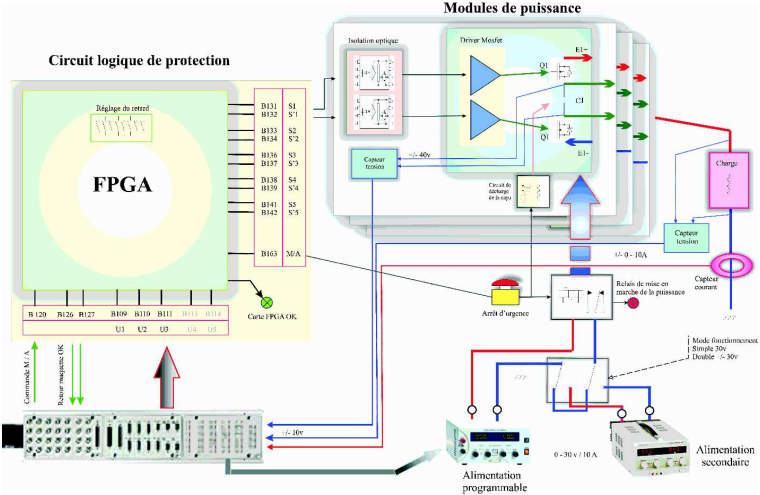


Figure: Three cell four level dc/dc converter

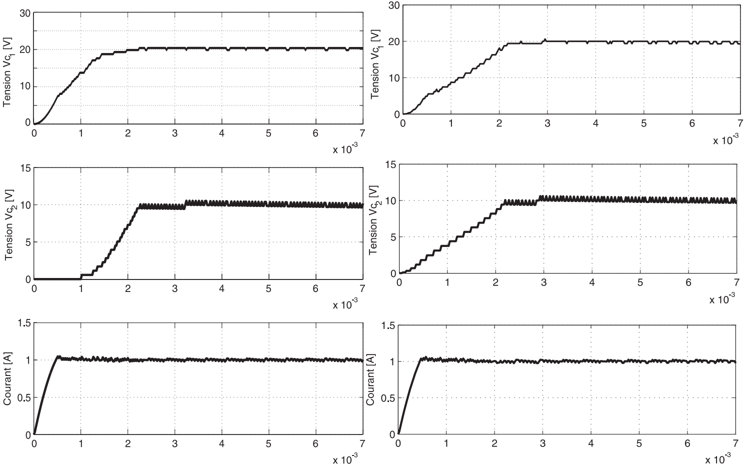
$$\begin{bmatrix} \dot{Vc_1} \\ \dot{Vc_2} \\ \dot{i_L} \end{bmatrix} = \begin{bmatrix} -\frac{i_L}{C_1} & \frac{i_L}{C_1} & 0 \\ 0 & -\frac{i_L}{C_2} & \frac{i_L}{C_2} \\ \frac{Vc_1}{L} & \frac{Vc_2 - Vc_1}{L} & \frac{E - Vc_2}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{R}{L} i_L \end{bmatrix} = g(x)u + f(x)$$

- $u_i \in \{0, 1\}$, $i \in \{1, 2, 3\}$.
- $C_1 = C_2 = 40\mu F$, $L = 10mH, R = 10\Omega$, $E = 30$ V.
- Average reference: $V_{c2}^{ref} = E/3$, $V_{c1}^{ref} = 2E/3$, $i_L^{ref} = 1A$

Experiment set-up



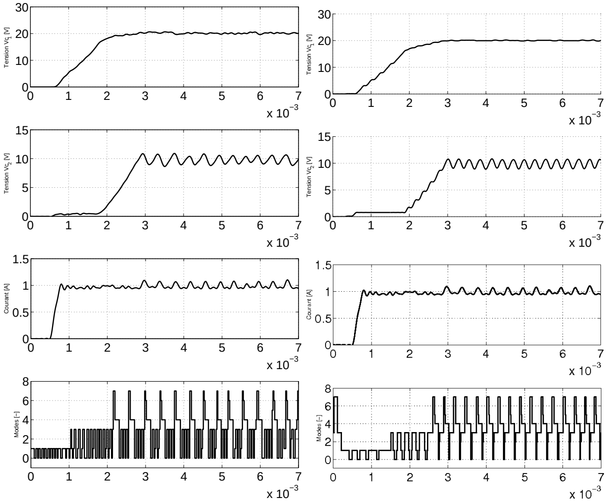
Simulation results (Start-up simulation response)



OPTIMAL CONTROL PREDICTIVE CONTROL

Sampling frequency 45kHz.

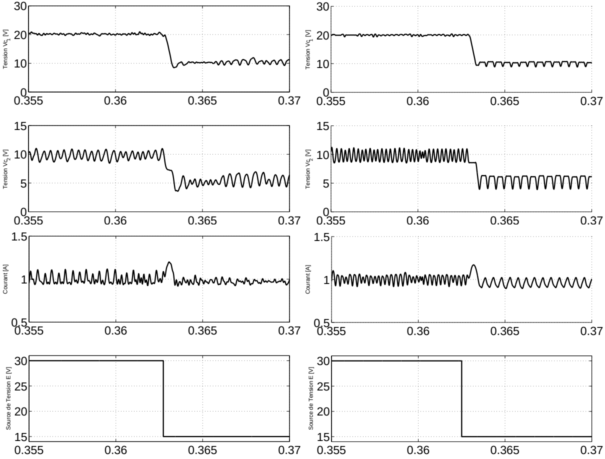
Experimental results: Optimal control (Start-up)



Experimental Simulation + delay

Sampling frequency 45kHz.

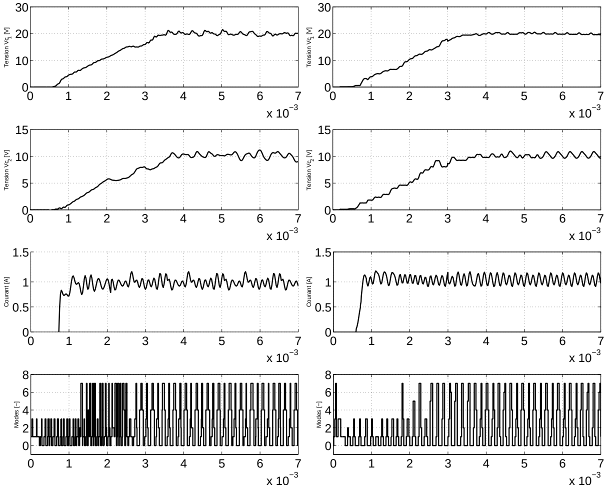
Experimental results: Optimal control ($E = 30V \Rightarrow 15V$)



Experimental Simulation + delay

Sampling frequency 45kHz.

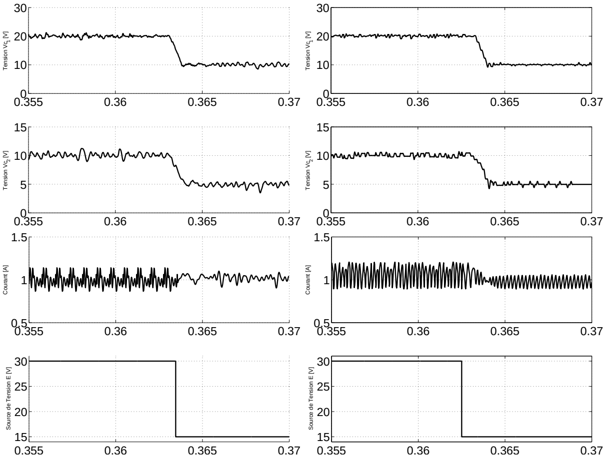
Experimental results: Predictive control (Start-up).



Experimental Simulation + delay

Sampling frequency 45kHz.

Experimental results: Predictive control ($E = 30V \Rightarrow 15V$).



Experimental Simulation + delay

Sampling frequency 45kHz.

Remarks

- Even if robusteness criteria are not taken into account, it seems to be obtained.
- Methods also applied in other power converters. Multilevel with other parameters, step-down (Simulation and experimentation) and Step-Up, SEPIC (Simulation)
- The sampling frequency is limited to 45 kHz. Higher frequency, better results.
- The mesures are affected by a delay. Question: Where is this delay coming from?

Conclusions

- Studied two new methods for cyclic switched systems.
- Transitory $\Rightarrow$ Optimal control and singular arcs.
- Optimal control results can be used as reference results (Comparaison with suboptimal strategies).
- A large off-line computing time, but a small on-line computing time (Interpolation).
- Steady state (Cycle) $\Rightarrow$ Predictive control (Cycle tracking).
- Experiments on real converters.

Perspectives

- Solve the delay problem (short time)
- Try to increase the cases to obtain the singular arcs [Meziat et al., 2007].
- Combine optimal and predictive control?.
- Constrains in the states.
- Take into account autonomous jumps in the model (Mode DCM for power converters).
- Study robusteness to the methods $\Rightarrow$ Adaptive control for the parameter estimation?.
- Observer for this kind of systems.
- Other criteria: Harmonics (Frequency criterion).
- Direct discrete approaches.

...THANK YOU FOR YOUR ATTENTION...

An easier form to solve is given by (independent of λ)

Proposition

If (x, u) is an extremal solution with $2q < n$. Then the (x, u)-locus satisfies

$$S(x, y_0, ..., y_{(n-2q-1)}) = 0$$

where $(y_0, ..., y_{(n-2q-1)}) = (u, \dot{u}, u^{(2)}, \ldots, u^{(n-2q-1)})$
and S is given by:

$$S = det\left(\left[g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), N_0, N_1, \ldots, N_{n-2q-1}\right]\right)$$

with

$$N_i(x, u) = ad_f^{2q+i} g(x) + \sum_{k=0}^{i} \left(I \frac{d}{dt} - f_x - u(t) g_x\right)^k ([g, ad_f^{2q-k} g](x) u(t))$$

$\frac{d}{dt}(*) = (f(x) + ug(x)) \frac{\partial(*)}{\partial x} + \frac{\partial(*)}{\partial t}$

An easier form to solve is given by (independent of λ)

Proposition

If (x, u) is an extremal solution with $2q < n$. Then the (x, u)-locus satisfies

$$S(x, y_0, ..., y_{(n-2q-1)}) = 0$$

where $(y_0, ..., y_{(n-2q-1)}) = (u, \dot{u}, u^{(2)}, \ldots, u^{(n-2q-1)})$
and S is given by:

$$S = det\left(\left[g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), N_0, N_1, \ldots, N_{n-2q-1}\right]\right)$$

with

$$N_i(x, u) = ad_f^{2q+i} g(x) + \sum_{k=0}^{i} \left(I\frac{d}{dt} - f_x - u(t)g_x\right)^k ([g, ad_f^{2q-k} g](x)u(t))$$

$\frac{d}{dt}(*) = (f(x) + ug(x))\frac{\partial(*)}{\partial x} + \frac{\partial(*)}{\partial t}$

Remark

Control u in function of x, $\dot{u}$, $u^{(2)}$, ...

$\Downarrow$

No unicity of the solution.

How to determine singular arcs? (Algebraically determined)

For free time optimal control problems:
$H = 0 \Rightarrow \lambda^T f(x) = 0$, then if $n = 2q + 1$:

1.

$$u(x) = -\frac{det([f(x), g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), ad_f^{2q} g(x)])}{det([f(x), g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), [g, ad_f^{2q-1} g](x)])}$$

2. Only $u(x) \in co(U)$ (since they can be "approximated" by $\{0, 1\}$).
3. λ can be a-posteriori deduced from:

$$\lambda^T [f(x), g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x)] = 0. \qquad (9)$$

4. Is the second order condition verified?

Remark: Problem order and arc order

Problem order: Smallest integer q such that appears explicitely in $\phi^{(2q)}$:

$$\phi^{(2q)} = H_u^{(2q)}(u, x, \lambda) = A(x, \lambda) + uB(x, \lambda) \qquad (10)$$

What happens if it exists a solution (x^*, λ^*) such that $B(x^*, \lambda^*) = 0$? $(A(x, \lambda) + \cancel{uB(x, \lambda)} = 0.)$

Definition

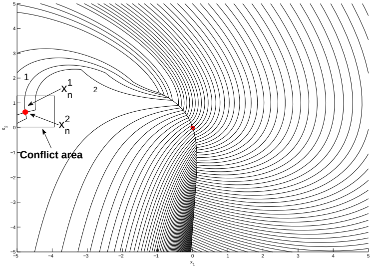
If (x^*, λ^*) is a singular arc defined in $[a, b] \subset T$. The **arc order** is the smallest integer p such that:

$$(\phi^{(2p)})_u = ((H_u)^{(2p)})_u(x^*, u^*) \neq 0 \qquad (11)$$

for all $t \in [a, b]$.

Optimal trajectories synthesis

Singular arc ⇒ Regular trajectories ⇒ Interpolation ⇒ Feedback control.



- x_n^i: Cost until the conflict point for the trajectory i.
- Which trajectory gives the minimum cost?: It stays!
- The others are cut and erased from the conflict point.

The trajectories are interpolated using a neural network (Offline)

⇓

State feedback control $u(x)$

Buisson, J., Cormerais, H., and Richard, P. (2002).
Analysis of the bond graph model of hybrid physical systems with ideal switches.
Journal of Systems and Control Engineering, 216(11):47–72.

Dmitruk, A. and Kaganovich, A. (2008).
The hybrid maximum principle is a consequence of pontryagin maximum principle.
Systems & Control Letters, 57:964–970.

Fraser-Andrew, G. (1989).
Finding candidate singular optimal controls: A state of the art survey.
Journal of Optimization Theory and Applications, 60(2):173 – 190.

Meziat, R., Patino, D., and Pedregal, P. (2007).
An alternative approach for non-linear optimal control problems based on the method of moments.
Comput. Optim. Appl., 38(1):147–171.

Pontryagin, L., Boltyanskii, V., Gamkrelidze, R., and Mishchenko, E. (1964).
The Mathematical Theory of Optimal Processes.
Pergamon.

Riedinger, P. and Iung, C. (1999).
Maximum principle and optimal control for hybrid systems: an hysteresis example.
Proceedings of IEEE Conference on System Man and Cybernetics, 1:188 – 193.

Riedinger, P., Iung, C., and Kratz, F. (2003).
An optimal control approach for hybrid systems.
European Journal of Control, 9(5):449–458.

Robbins, H. (1967).
A generalized legendre-clebsch-condition for the singular cases of optimal control.
IBM Journal of Research and Development, 11(4):361–372.

Sanders, J. A. and Verhulst, F. (1985).
Averaging Methods in Nonlinear Dynamical Systems.
Springer-Verlag.

Shaikh, M. and Caines, P. (2003).
On the optimal control of hybrid systems: Analysis and algorithms for trajectory and schedule optimization.
In *Proceedings of the IEEE Conference on Decision and Control*, volume 3, pages 2144– 2149, Acapulco, Mexico.

Sussman, H. (1999).
A maximum principle for hybrid optimal control problems.
In *Proceedings of the 38nd IEEE Conference on Decision and Control*, volume 1, pages 425 – 430, Maui, Hawaii.

Volker, M. (1996).
Singular optimal control - the state of the art.
CAS-29.