Nonlinear Model Predictive Control
More than an introduction...

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Atelier technique
Regardless of the control strategy being used, the following keywords has to be addressed:

- **System model**
  (State, control, measurement, disturbance)

- **Constraints**
  (on state, control)

- **Performance index**
  (Operational cost, energy consumption, tracking quality)

- **Stability**

- **Robustness**
Current state

Desired state
Nonlinear Model Predictive Control

Current state

Desired state

State constraint

Current state
Nonlinear Model Predictive Control

Current state
Desired state
Performance index:
\[ \text{Length of the steering path} \]
At decision instant $k = 0$, the optimal solution is $S_0$.
Current state

Desired state

Go on one step ahead …
Measure the present situation & Compute a new optimal solution
Go on one step ahead
Measure the present situation & Compute a new optimal solution
go on one step ahead
Measure the present situation & Compute a new optimal solution
Go on one step ahead
Nonlinear Model Predictive Control

- Desired state
- Initially computed trajectory
- Closed loop trajectory

M. Alamir (–)
A simple feedback principle (informal)

- At each decision instant, evaluate the situation
A simple feedback principle (informal)

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- Based on the evaluation, compute the best strategy
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- Based on the evaluation, compute the best strategy
- Apply the beginning of the strategy until the next decision instant
A simple feedback principle (informal)

- At each decision instant, evaluate the situation
- Based on the evaluation, compute the best strategy
- Apply the beginning of the strategy until the next decision instant
- Re-evaluate the situation
A simple feedback principle (informal)

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- Apply the beginning of the strategy until the next decision instant
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- Recompute the best strategy
A simple feedback principle (informal)

- At each decision instant, evaluate the situation
- Based on the evaluation, compute the best strategy
- Apply the beginning of the strategy until the next decision instant
- Re-evaluate the situation
- Recompute the best strategy
- Apply the first part until the next decision instant
A simple feedback principle (informal)

- At each decision instant, evaluate the situation
- Based on the evaluation, compute the best strategy
- Apply the beginning of the strategy until the next decision instant
- Re-evaluate the situation
- Recompute the best strategy
- Apply the first part until the next decision instant
- Keep doing
A simple feedback principle (Formal)

- At decision instant $k$, measure the state $x(k)$
A simple feedback principle (Formal)

- At decision instant $k$, measure the state $x(k)$
- Based on $x(k)$, compute the best sequence of actions:
  \[ u^0(x(k)) := (u^0(k; x(k)) \ u^0(k + 1; x(k)) \ \ldots \ u^0(k + i; x(k)) \ \ldots) \]
A simple feedback principle (Formal)

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- Apply the control $u^0(k; x(k))$ on the sampling period $[k, k + 1]$
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- At decision instant $k + 1$, measure the state $x(k + 1)$
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- At decision instant $k + 1$, measure the state $x(k + 1)$
- Based on $x(k + 1)$, compute the best sequence of actions:
  $$u^0(x(k + 1)) := (u^0(k + 1; x(k + 1)) \quad u^0(k + 2; x(k + 1)) \quad \ldots)$$
A simple feedback principle (Formal)

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  $$u^0(x(k + 1)) := (u^0(k + 1; x(k + 1)) \ u^0(k + 2; x(k + 1)) \ldots)$$
- Apply the control $u^0(k + 1; x(k + 1))$ on the sampling period $[k + 1, k + 2]$
A simple feedback principle (Formal)

- At decision instant $k$, measure the state $x(k)$
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- Apply the control $u^0(k + 1; x(k + 1))$ on the sampling period $[k + 1, k + 2]$
- $\ldots$
A sampled state feedback

- **At decision instant** $k$, measure the state $x(k)$
- Based on $x(k)$, compute the best sequence of actions:
  \[
  u^0(x(k)) := (u^0(k; x(k)) \ u^0(k + 1; x(k)) \ldots \ u^0(k + i; x(k)) \ldots)
  \]
- **Apply the control** $u^0(k; x(k))$ on the sampling period $[k, k + 1]$
- At decision instant $k + 1$, measure the state $x(k + 1)$
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  \]
- **Apply the control** $u^0(k + 1; x(k + 1))$ on the sampling period $[k + 1, k + 2]$
- ...
A sampled state feedback

- **At decision instant** $k$, measure the state $x(k)$
- Based on $x(k)$, compute the best sequence of actions:
  $$u^0(x(k)) := (u^0(k; x(k)) \; u^0(k + 1; x(k)) \; \ldots \; u^0(k + i; x(k)) \; \ldots)$$
- **Apply the control** $u^0(k; x(k))$ **on the sampling period** $[k, k + 1]$

A state feedback

We have defined a sampled state feedback

$$u(k) = u^0(k; x(k))$$
A key task ...

- At decision instant $k$, measure the state $x(k)$
- Based on $x(k)$, **compute the best sequence of actions**:
  $$u^0(x(k)) := (u^0(k; x(k)) \ u^0(k + 1; x(k)) \ldots \ u^0(k + i; x(k)) \ldots)$$
- Apply the control $u^0(k; x(k))$ on the sampling period $[k, k + 1]$
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  $$u^0(x(k + 1)) := (u^0(k + 1; x(k + 1)) \ u^0(k + 2; x(k + 1)) \ldots)$$
- Apply the control $u^0(k + 1; x(k + 1))$ on the sampling period $[k + 1, k + 2]$
- ...
A key task . . .

- At decision instant $k$, measure the state $x(k)$
- Based on $x(k)$, compute the best sequence of actions:
  $$u^0(x(k)) := (u^0(k; x(k)) \ u^0(k + 1; x(k)) \ ... \ u^0(k + i; x(k)) \ ...)$$
- Apply the control $u^0(k; x(k))$ on the sampling period $[k, k + 1]$

We need an optimization problem

$$\mathcal{P}(x(k)) : \min_u \left\{ V(x(k), u) \mid u \in \mathcal{U}(x(k)) \right\}$$

$u^0(x(k))$ is a solution of $\mathcal{P}(x(k))$
How to define & Solve $\mathcal{P}(x)$ for our example?
Step 1 : Write down the system model

State of the system: \( x = \begin{pmatrix} x_1 \\ x_2 \\ \theta \\ V \end{pmatrix} \)

Desired state

\[ V = \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \]
Step 1: Write down the system model

State of the system: \( x = \begin{pmatrix} x_1 \\ x_2 \\ \theta \\ V \end{pmatrix} \)

\[
\begin{align*}
\dot{x}_1 &= x_4 \cos(x_3) \\
\dot{x}_2 &= x_4 \sin(x_3) \\
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= u_2
\end{align*}
\]

\[ V = \sqrt{\dot{x}_1^2 + \dot{x}_2^2} \]
Step 1: Write down the system model

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\dot{x}_3 & = u_1 \\
\dot{x}_4 & = u_2
\end{align*}
\]

\[V = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\]

function xdot=car(t,x)
    global u_glob
    xdot=zeros(4,1);
    xdot(1)=x(4)*cos(x(3));
    xdot(2)=x(4)*sin(x(3));
    xdot(3)=u_glob(1);
    xdot(4)=u_glob(2);
    return
Step 2 : Obtain the sampled-time model

Given the continuous system

\[ \dot{x} = f_c(x, u) \]

Compute the implicit \( \tau \)-discrete dynamics

\[ x^+(k) = x(k + 1) = f(x(k), u(k)) \]

where \( x(k + 1) \) is the solution at instant \( \tau \) of

\[ \dot{\xi} = f_c(\xi, u(k)) \quad ; \quad \xi(0) = x(k) \]
function xdot=car(t,x)
    global u_glob
    xdot=zeros(4,1);
    xdot(1)=x(4)*cos(x(3));
    xdot(2)=x(4)*sin(x(3));
    xdot(3)=u_glob(1);
    xdot(4)=u_glob(2);
    return

function xplus=car_d(x,u,tau)
    global u_glob
    u_glob=u;
    [tt,xx]=ode45(@car,[0,tau],x);
    xplus=xx(end,:);
    return
Some notations before we continue

Consider

\[ x^+ = f(x, u) \]

\[ u := (u(0), u(1), \ldots, u(N-1)) \]

Notation

\[ x^u(\cdot, x(k)) := \left\{ x(k + i) \right\}_{i=0}^{N} \]

\[ x(k+i) = f(x(k+i-1), u(i-1)) \]
Recall

\[ P(x(k)) : \min_u \left\{ V(x(k), u) \mid u \in U(x(k)) \right\} \]

\[ u^0(x(k)) \quad \text{is a solution of} \ P(x(k)) \]
Step 3 : Write down the constraints
Step 3 : Write down the constraints

The final constraint

\[ x^u(N; x(k)) - x^d = 0 \]

Obstacle avoidance

\[ r - \min_{i \in \{1,2\}} d(x^u(\cdot; x(k)), c_i) \leq 0 \]

Saturation constraint

\[ u \in [u_{min}, u_{max}]^N \]

Desired state

\[ x^d := \begin{pmatrix} x^d_1 \\ x^d_2 \\ 0 \end{pmatrix} \]

Obstacle avoidance

\[ r = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} \]

Saturation constraint
Step 3 : Write down the constraints

Equality constraints

\[ G_1(u, x(k)) = 0 \]

Inequality constraints

\[ G_2(u, x(k)) \leq 0 \]

\[ U(x(k)) := \{ u \mid G_1 = 0 \text{ and } G_2 \leq 0 \} \]
Step 3 : Write down the constraints

function [G1,G2]=constraintes(les_u,x,tau)
global xd xi eta u_min u_max
N=length(les_u(:,1));
xu=zeros(N,4); xu(1,:)=x';
d(1)=min(norm(xu(1,1:2)-xi),norm(xu(1,1:2)-eta));
for i=2:N,
    xu(i,:)=car_d(xu(i-1,:),lesu(i-1,:),tau);
    d(i)=min(norm(xu(i,1:2)-xi),norm(xu(i,1:2)-eta));
end
cond_min=max(max(ones(N,1)*u_min'-lesu));
cond_max=max(max(lesu-ones(N,1)*u_max'));
G1=xu(N,:)-xd;
G2=[r-min(d);cond_min;cond_max];
return
Recall

$$\mathcal{P}(x(k)) : \min_u \left\{ V(x(k), u) \mid u \in \mathcal{U}(x(k)) \right\}$$

$$u^0(x(k))$$ is a solution of $$\mathcal{P}(x(k))$$
Write down the performance index

\[ V(x(k), u) := \sum_{i=1}^{N} \| x^u(i; x(k)) - x^d \|^2 \]
Write down the performance index

\[ V(x(k), u) := \sum_{i=1}^{N} \| x^u(i; x(k)) - x^d \|^2 \]

function f=cost(lesu,x,tau)
    global xd
    N=length(les_u(:,1));
    xu=zeros(N,4);
    xu(1,:)=x';
    f = 0;
    for i=2:N,
        xu(i,:)=car_d(xu(i-1,:),lesu(i-1,:),tau);
        f = f+norm(xu(i,:)'-xd)^2;
    end
return
FMINCON finds a constrained minimum of a function of several variables.

FMINCON attempts to solve problems of the form:

\[
\begin{align*}
\min & \quad F(X) \\
\text{subject to:} & \quad A\cdot X \leq B, \quad A_{eq}\cdot X = B_{eq} \quad \text{(linear constraints)} \\
& \quad C(X) \leq 0, \quad C_{eq}(X) = 0 \quad \text{(nonlinear constraints)} \\
& \quad LB \leq X \leq UB
\end{align*}
\]

\(X=FMINCON(FUN,X0,A,B)\) starts at \(X0\) and finds a minimum \(X\) to the function \(FUN\), subject to the linear inequalities \(A\cdot X \leq B\). \(FUN\) accepts input \(X\) and returns a scalar function value \(F\) evaluated at \(X\). \(X0\) may be a scalar, vector, or matrix.

\(X=FMINCON(FUN,X0,A,B,A_{eq},B_{eq})\) minimizes \(FUN\) subject to the linear equalities \(A_{eq}\cdot X = B_{eq}\) as well as \(A\cdot X \leq B\). (Set \(A=[]\) and \(B=[]\) if no inequalities exist.)

\(X=FMINCON(FUN,X0,A,B,A_{eq},B_{eq},LB,UB)\) defines a set of lower and upper bounds on the design variables, \(X\), so that a solution is found in the range \(LB \leq X \leq UB\). Use empty matrices for \(LB\) and \(UB\) if no bounds exist. Set \(LB(i) = -\infty\) if \(X(i)\) is unbounded below; set \(UB(i) = \infty\) if \(X(i)\) is unbounded above.

\(X=FMINCON(FUN,X0,A,B,A_{eq},B_{eq},LB,UB,NONLCON)\) subjects the minimization to the constraints defined in \(NONLCON\). The function \(NONLCON\) accepts \(X\) and returns the vectors \(C\) and \(C_{eq}\), representing the nonlinear inequalities and equalities respectively. \(FMINCON\) minimizes \(FUN\) such that \(C(X) \leq 0\) and \(C_{eq}(X) = 0\). (Set \(LB=[]\) and/or \(UB=[]\) if no bounds exist.)
Coming next . . .

- Existence of solutions
- Closed-loop stability
Existence of solutions

\[ P(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in \mathbb{X} \quad \text{and} \quad x^u(N, x) \in X_f \right\} \]

in which \( \mathbb{U} \) compact ; \( \mathbb{X} \) compact ; \( X_f = \{ x^d \} \subset \mathbb{X} \) closed.
Existence of solutions

\[ \mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} \]

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in which \( \mathcal{U} \) compact; \( \mathcal{X} \) compact; \( X_f = \{ x^d \} \subset \mathcal{X} \) closed.

No global definition under bounded control

one must have \( x \in X_N \) the subset of states from which \( X_f \) is accessible by \( N \)-step controls in \( \mathcal{U} \) and trajectories in \( \mathcal{X} \).
Existence of solutions

\[ \mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in \mathbb{X} \quad \text{and} \quad x^u(N, x) \in X_f \right\} \]

in which \( \mathbb{U} \) compact; \( \mathbb{X} \) compact; \( X_f = \{ x^d \} \subset \mathbb{X} \) closed.

for all \( x \in X_N \), main arguments

\( V(x, \cdot) \) is continuous and \( \mathcal{U}(x) \) is compact.
Close-loop stability

\[ \mathcal{P}(x) : \min_{u} \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in \mathbb{X} \quad \text{and} \quad x^u(N, x) \in X_f \right\} \]

\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

\[ x^d = 0 \quad ; \quad X_f = \{0\} \quad ; \quad f(0, 0) = 0 \]
Close-loop stability

\[
\mathcal{P}(x) : \min_u \left\{ V(x,u) \mid u \in \mathcal{U}(x) \right\}
\]

\[
\mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in \mathbb{X} \quad \text{and} \quad x^u(N, x) \in X_f \right\}
\]

\[
V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i))
\]

\[
x^d = 0 \quad ; \quad X_f = \{0\} \quad ; \quad f(0,0) = 0
\]

An optimal solution :

\[
u^0(x) := (u^0(0; x) \quad u^0(1; x) \quad \ldots \quad u^0(N - 1; x)) \in \mathbb{U}^N
\]
Close-loop stability

\[ \mathcal{P}(x) : \min_{u} \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in X \quad \text{and} \quad x^u(N, x) \in X_f \right\} \]

\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

\[ x^d = 0 \quad ; \quad X_f = \{0\} \quad ; \quad f(0, 0) = 0 \]

An optimal solution:

\[ u^0(x) := (u^0(0; x) \quad u^0(1; x) \quad \ldots \quad u^0(N - 1; x)) \in \mathbb{U}^N \]

Sampled state feedback:

\[ \kappa^0(x) := u^0(0; x) \]
Study the stability of

\[ x^+ = f(x, u^0(0; x)) \]
$L(x^u(\cdot; x), u(\cdot))$

$u^0(x)$ solution of $\mathcal{P}(x)$

$u^0(0; x)$

$u^0(N - 1; x)$
\[ L(x^u(\cdot; x), u(\cdot)) \]

\[ x^+ = f(x, u^0(0; x)) \]
Optimal solution of $P(x^+)$?
An admissible candidate sequence $\tilde{u}(\cdot, x^+) \in \mathcal{U}(x^+)$
An admissible candidate sequence $\tilde{u}(\cdot, x^+) \in \mathcal{U}(x^+)$
$L(x^u(\cdot; x), u(\cdot))$

$x^+ = f(x, u^0(0; x))$

$V^0(x^+) \leq V(x^+, \tilde{u})$ (definition of optimality)
\[ L(x^u(\cdot; x), u(\cdot)) \]

\[ x^+ = f(x, u^0(0; x)) \]

\[ V^0(x^+) \leq V(x^+, \tilde{u}) \leq V^0(x) - L(x, u^0(0; x)) \]
Closed loop stability

Therefore, on the closed loop trajectory:

\[ V^0(x(k+1)) \leq V^0(x(k)) - L(x(k), \kappa^0(x(k))) \]

therefore,

\[ \lim_{k \to \infty} L(x(k), \kappa^0(x(k))) = 0 \]

Consequently, if \( L(\cdot, \cdot) \) is a continuous positive definite function in \( x \), one has

\[ \lim_{k \to \infty} x(k) = 0 \]
Close-loop stability

\[ \mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \ x^u(i; x) \in \mathbb{X} \text{ and } x^u(N, x) \in X_f \right\} \]

\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

\[ x^d = 0 \quad ; \quad X_f = \{0\} \quad ; \quad f(0, 0) = 0 \]

The final equality constraint : A key role
Nonlinear Model Predictive Control

\[ L(x^u(\cdot; x), u(\cdot)) \]

\[ x^+ = f(x, u^0(0; x)) \]

Optimal solution of \( P(x^+) \)?
\[ L(x^u(\cdot; x), u(\cdot)) \]

\[ x^+ = f(x, u^0(0; x)) \]

An admissible candidate sequence \( \tilde{u}(\cdot, x^+) \in \mathcal{U}(x^+) \)

\[ f(0, 0) = 0 \]
First useful property of $X_f = \{0\}$

For all $\xi \in X_f$, there exists an admissible control $\kappa^f(\xi)$ such that:

$$f(\xi, \kappa^f(\xi)) \in X_f$$

($X_f$ is positively invariant under $\kappa^f(\cdot)$)
\[ L(x^{u}(\cdot; x), u(\cdot)) \]

\[ x^+ = f(x, u^0(0; x)) \]

\[ V^0(x^+) \leq V(x^+, \tilde{u}) \leq V^0(x) - L(x, u^0(0; x)) \]
Nonlinear Model Predictive Control

\[ L(x^{u}(\cdot; x), u(\cdot)) \quad T_2 = F(f(\xi, \kappa^f(\xi))) + L(\xi, \kappa^f(\xi)) \]

\[ x^+ = f(x, u^0(0; x)) \quad T_1 = F(\xi) \]

\[ V^0(x^+) \leq V(x^+, \tilde{u}) = V^0(x) - L(x, u^0(0; x)) + (T_2 - T_1) \]
\[
L(x^u(\cdot; x), u(\cdot))
\]

\[
x^+ = f(x, u^0(0; x))
\]

\[
T_2 = F(f(\xi, \kappa^f(\xi))) + L(\xi, \kappa^f(\xi))
\]

\[
T_1 = F(\xi)
\]

\[
V^0(x^+) \leq V(x^+, \tilde{u}) = V^0(x) - L(x, u^0(0; x)) + \left( T_2 - T_1 \right) \leq 0
\]

\[
F(f(\xi, \kappa^f(\xi)) - F(\xi) \leq -L(\xi, \kappa^f(\xi))
\]
Second useful property of $X_f = \{0\}$

For all $\xi \in X_f$, the admissible control $\kappa^f(\xi)$ is such that:

$$F(f(\xi, \kappa^f(\xi)) - F(\xi) \leq -L(\xi, \kappa^f(\xi))$$

(The terminal cost $F(\cdot)$ is a Lyapunov function for the closed loop dynamics under $\kappa^f(\cdot)$)
First useful property of \( X_f = \{0\} \)

For all \( \xi \in X_f \), there exists an admissible control \( \kappa^f(\xi) \) such that:

\[
f(\xi, \kappa^f(\xi)) \in X_f
\]

\( (X_f \text{ is positively invariant under } \kappa^f(\cdot)) \)

Second useful property of \( X_f = \{0\} \)

For all \( \xi \in X_f \), the admissible control \( \kappa^f(\xi) \) is such that:

\[
F(f(\xi, \kappa^f(\xi)) - F(\xi) \leq -L(\xi, \kappa^f(\xi))
\]

\( (\text{The terminal cost } F(\cdot) \text{ is a Lyapunov function for the closed loop dynamics under } \kappa^f(\cdot)) \)
Necessary conditions for well defined and stable NMPC scheme

\[ P(x) : \min_{u} \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} ; \quad x^+ = f(x, u) \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in X \quad \text{and} \quad x^u(N, x) \in X_f \right\} \]

\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]
Necessary conditions for well defined and stable NMPC scheme

\[ \mathcal{P}(x) : \min_{u} \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} ; \quad x^+ = f(x, u) \]

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\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

Condition 1 : Continuity

The applications \( f, F, L \) are continuous in their arguments.
Necessary conditions for well defined and stable NMPC scheme

\[ \mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} ; \quad x^+ = f(x, u) \]

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\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

Condition 2 : Compactness

✓ \( \mathbb{U} \) is compact

✓ \( \mathbb{X} \) and \( X_f \subset \mathbb{X} \) are closed.
Necessary conditions for well defined and stable NMPC scheme

\[ \mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} ; \quad x^+ = f(x, u) \]

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\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

Condition 3 : Detectability

The integral cost \( L \) must be such that

\[ \left\{ L(x, u) \to 0 \right\} \Rightarrow \left\{ x \to 0 \right\} \]
Necessary conditions for well defined and stable NMPC scheme

\[
\mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} ; \quad x^+ = f(x, u)
\]

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\]

\[
V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i))
\]

Condition 4 : \(X_f\) is positively invariant under some local \(\kappa^f(\cdot)\)

For all \(\xi \in X_f\), there exists an admissible control \(\kappa^f(\xi)\) such that :

\[
f(\xi, \kappa^f(\xi)) \in X_f
\]
Nonlinear Model Predictive Control

Necessary conditions for well defined and stable NMPC scheme

\[\mathcal{P}(x) : \min_u \left\{ V(x, u) \mid u \in \mathcal{U}(x) \right\} ; \quad x^+ = f(x, u)\]

\[\mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \quad x^u(i; x) \in \mathbb{X} \quad \text{and} \quad x^u(N, x) \in X_f \right\}\]

\[V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i))\]

Condition 5: The terminal cost \(F(\cdot)\) is a Lyapunov under \(\kappa^f(\cdot)\)

For all \(\xi \in X_f\), the admissible control \(\kappa^f(\xi)\) is such that:

\[F(f(\xi, \kappa^f(\xi)) - F(\xi) \leq -L(\xi, \kappa^f(\xi))\]
Nonlinear Model Predictive Control

Necessary conditions for well defined and stable NMPC scheme

\[ \mathcal{P}(x) : \min_{u} \mathcal{V}(x, u) \mid u \in \mathcal{U}(x) \] ; \[ x^+ = f(x, u) \]

\[ \mathcal{U}(x) := \left\{ u \in \mathbb{U}^N \mid \forall i \; x^u(i; x) \in \mathbb{X} \; \text{and} \; x^u(N, x) \in X_f \right\} \]

\[ V(x, u) = F(x(N; x)) + \sum_{i=0}^{N} L(x^u(i; x), u(i)) \]

Condition 6 : Feasibility

There is at least a sequence that meets the constraints (in particular \( x(0) \in X_N \) (the subset of state steerable in \( N \) steps to \( X_f \) with bounded controls in \( \mathbb{U} \)).
To summarize

The system \( x^+ = f(x, u) \)

The cost function \( F(x(N)) + \sum_{i=0}^{N} L(x(i), u(i)) \)

1) \( f, L, L \) are continuous.
2) \( \mathbb{U} \) is compact, \( \mathbb{X} \) and \( X_f \subset \mathbb{X} \) are closed.
3) \( \{L(x, u) \to 0\} \Rightarrow \{x \to 0\} \)
4) \( X_f \) is positively invariant (with some \( \kappa_f(\cdot) \).
5) \( F(\cdot) \) is a Lyapunov function under \( \kappa_f(\cdot) \).
6) \( x(0) \in X_N \).
How to choose $X_f$ and $\kappa_f(\cdot)$?

- **Infinite horizon** $N = \infty$.
How to choose $X_f$ and $\kappa_f(\cdot)$?

1. Infinite horizon $N = \infty$.
2. Point-wise final constraint $X_f = \{0\}$.

If the linearized system at $0$ is stabilizable, then

$$\kappa_f(x) = -Kx(A - BK)^T P (A - BK) - P = -Q$$

for some $P, Q > 0$. $X_f$ is a level set of $x^T Px$, namely

$$X_f = \{x | x^T Px \leq \varrho\}$$

for a sufficiently small $\varrho > 0$. 
How to choose $X_f$ and $\kappa_f(\cdot)$?

1. Infinite horizon $N = \infty$.
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How to choose $X_f$ and $\kappa_f(\cdot)$?

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How to choose $X_f$ and $\kappa_f(\cdot)$?

1. Infinite horizon $N = \infty$.
2. Point-wise final constraint $X_f = \{0\}$.
3. If the linearized system at $0$ is stabilizable, then
   - $\kappa_f(x) = -Kx$
   - $(A - BK)^TP(A - BK) - P = -Q$ for some $P, Q > 0$
   - $X_f$ is a level set of $x^TPx$, namely
     $$X_f = \left\{ x \mid x^TPx \leq \varrho \right\}$$
     for a sufficiently small $\varrho > 0$. 
Example

\[ x_1^+ = x_1 + (1 + x_2^2)u \]
\[ x_2^+ = \frac{3}{2}x_2 - x_1e^u \]

- Open-loop instable.
- The set of equilibrium states is given by

\[ \mathcal{E}_{st} = \left\{ x^{(\alpha)} := \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} ; \quad \alpha \in \mathbb{R} \right\} \]

Control objective starting at \( x^{(0)} = (0, 0) \), stabilize the system around \( x^{(1)} = (1, 2) \).
Consider the cost function

\[ V(x, u) := F(x(N)) + \sum_{i=0}^{N} L(x(i), u(i)) \]

where

\[ L(x, u) := \| x - x^{(1)} \|^2 + ru^2 \]

\[ F(x) = \sum_{i=1}^{M-N} x^0(i; x) \]
Test 1: $N = 2, M = 2, r = 1$
Test 1: $N = 2, M = 2, r = 1$
Let us assume that one looks for solution such that \( u \leq 0.1 \).

Assume that for that reason one takes

\[ r = 160 \]
Test 2: $N = 2, M = 2, r = 160$
Test 2 : $N = 2, \ M = 2, \ r = 160$
Let us take $M = 3$ (instead of 2)
Test 3: $N = 2, M = 3, r = 160$
Test 3: $N = 2$, $M = 3$, $r = 160$
Increasing $M$ enabled the target state to be reached.

However, the control is again above 0.1,

So let us increase $r$ again by taking

$$r = 500$$
Test 4: \( N = 2, \ M = 3, \ r = 500 \)
Test 4: $N = 2, M = 3, r = 500$
A limit cycle appears

and the control is beyond 0.25!!!
Let us explicitly impose the constraint

\[ \mathbb{U} = [-0.1, 0.1] \]

and test it with the configuration

\[ r = 1 \quad ; \quad N = M = 3 \]
Test 5: $N = 3, M = 3, r = 1$, explicit constraint $|u| \leq 0.1$
Test 5: $N = 3$, $M = 3$, $r = 1$, explicit constraint $|u| \leq 0.1$
The example shows that

- Nonlinear Model Predictive Control is a "generic" solution.

(Who remember the system’s equations?!?)
subroutine syst(x,u,xplus)
    implicit none
    double precision :: x(2), u, xplus(2)
    xplus(1)=x(1)+(1.0d0+x(2)**2)*u
    xplus(2)=1.5*x(2)-x(1)*dexp(u)
end subroutine syst
subroutine syst_long(x0, nu, ns, utilde, xtilde)
  implicit none
  external syst
  integer :: nu, ns, i
  double precision :: x0(2), utilde(nu), xtilde(ns+1,2)
  xtilde(1,:) = x0
  do i = 1, nu
    call syst(xtilde(i,:), utilde(i), xtilde(i+1,:))
  enddo
  do i = nu+1, ns
    call syst(xtilde(i,:), 0.0d0, xtilde(i+1,:))
  enddo
end subroutine syst_long
subroutine criterie(nu,utilde,J)
    use param
    use imsl
    implicit none
    integer :: nu, i
    double precision :: utilde(nu), J, xtilde(M+1,2)
    double precision :: dx(2)
    call syst_long(x0_glob,nu,M,utilde,xtilde)
    J=0.d0
    do i=1,nu
        J=J+qu*dabs(utilde(i))*2
        dx = xtilde(i+1,:) - xd_glob
        J = J + qx*dot_product(dx, eye(2).x.dx)
    enddo
    do i=nu+1,M
        dx = xtilde(i+1,:) - xd_glob
        J = J + qx*dot_product(dx, eye(2).x.dx)
    enddo
end subroutine criterie
Computation of the optimal sequence

```fortran
subroutine uhat(x0, xd, nu, utilde, J)
    use param
    implicit none
    external criterere
    integer :: nu, maxfcn=100
    double precision :: x0(2), xd(2), utilde(nu), J
    double precision :: utilde_guess(nu), ulb(nu), uub(nu)
    x0_glob=x0; xd_glob=xd; utilde_guess=0.0d0
    ulb=-umax; uub=umax
    call dbcpol(criterere,nu,utilde_guess,0,ulb,uub,1.0d-8,maxfcn,utilde,J)
end subroutine uhat
```
The example shows that

- Nonlinear Model Predictive Control is a "generic" solution.
The example shows that

- Nonlinear Model Predictive Control is a "generic" solution.
- Easy handling of constraints
The example shows that

- Nonlinear Model Predictive Control is a "generic" solution.
- Easy handling of constraints
- The stability IS AN ISSUE
To summarize

The system $x^+ = f(x, u)$

The cost function $F(x(N)) + \sum_{i=0}^{N} L(x(i), u(i))$

1) $f$, $L$, $L$ are continuous.
2) $\mathcal{U}$ is compact, $\mathbb{X}$ and $X_f \subset \mathbb{X}$ are closed.
3) $\{L(x, u) \to 0\} \Rightarrow \{x \to 0\}$
4) $X_f$ is positively invariant (with some $\kappa_f(\cdot)$).
5) $F(\cdot)$ is a Lyapunov function under $\kappa_f(\cdot)$.
6) $x(0) \in X_N$. 
This general result summarizes 90% of existing works on the stability of the NMPC schemes.

This does not include the contractive schemes ...
A PRELIMINARY EXAMPLE

Consider the nonlinear system

\[
\dot{x}_1 = x_2 ; \quad \dot{x}_2 = x_1 u
\]

and the "candidate" Lyapunov function

\[
V(x) = \frac{1}{2} \left[ x_1^2 + x_2^2 \right]
\]

Compute the derivative of \( V \)

\[
\dot{V}(x) = x_1 x_2 (1 + u)
\]
A PRELIMINARY EXAMPLE

Consider the nonlinear system

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For classical Lyapunov design

\( V \) is not a good choice
(singular surface \( x_1 x_2 = 0 \))
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But...

For classical Lyapunov design

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But...
**Nonlinear Model Predictive Control**

**The basic idea: The contraction property**

\[
\min_{u(\cdot) \equiv u_{0} \in [-u_{\text{max}}, +u_{\text{max}}]} \| x_u(T, x_0(\theta)) \|
\]

\[\theta \in [0, \pi]\]

To summarize: A long term contraction property

\[\text{Whatever is the initial state } x_{0} \in B(0, 1), \text{ there exists constant control } u \in [-5, +5] \text{ such that } V(x_u(2, x_0)) \leq 0, V(x_0) \geq 0 \text{ or } V(x_u(4, x_0)) \leq 0.\]
Nonlinear Model Predictive Control

The basic idea: The contraction property

\[ \min_{u(\cdot) \equiv u_0 \in [-u_{max}, +u_{max}]} \| x_u(T, x_0(\theta)) \| \]

To summarize: A long term contraction property

Whatever is the initial state \( x_0 \in B(0, 1) \), there exists a constant control \( u_0 \in [-5, +5] \) such that

\[ V(x_u(2, x_0)) \leq 0 \]

or

\[ V(x_u(4, x_0)) \leq 0 \]
To summarize: A long term contraction property

Whatever is the initial state \( x_0 \in B(0, 1) \), there exists constant control \( u \in [-5, +5] \) such that

\[
V(x^u(2, x_0)) \leq 0.9 \, V(x_0) \quad \text{or} \quad [V(x^u(4, x_0)) \leq 0.82 \, V(x_0)]
\]
A bad formulation

Define a receding horizon feedback based on the following open-loop optimization problem

\[
\min_{u(\cdot), \Delta \in [0, T]} \int_0^{\Delta} L\left(x_u(\tau, x(t))\right) d\tau \quad \text{under} \quad V(x_u(t + \Delta, x(t))) \leq \gamma V(x(t))
\]
A bad formulation

Define a receding horizon feedback based on the following open-loop optimization problem

\[
\min_{u(\cdot), \Delta \in [0,T]} \int_0^\Delta L(x_u(\tau, x(t))) \, d\tau \quad \text{under} \quad V(x_u(t + \Delta, x(t))) \leq \gamma V(x(t))
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\]

Updating systematically the contractive constraint

\[V(x_u(\Delta, x(t))) \leq \gamma V(x(t))\]

May cause instability in closed-loop
Nonlinear Model Predictive Control

Using the contraction property in feedback design

Classical contractive Receding horizon schemes

Either

Use the open-loop control

\[ \hat{u}(\cdot, x(t)) \]

on \([t, t + \hat{\Delta}(x(t))]\)
Or

Memorize $x(t)$ and use

$$V(x_u(\Delta, x(t + k\tau_s))) \leq \gamma V(x(t))$$

in a RH scheme during the time interval $[t, t + \hat{\Delta}(x(t))]$
Nonlinear Model Predictive Control

Using the contraction property in feedback design

Classical contractive Receding horizon schemes


Or

Memorize $x(t)$ and use

$$V(x_u(\Delta, x(t + k\tau_s))) \leq \gamma V(x(t))$$

in a RH scheme during the time interval $[t, t + \hat{\Delta}(x(t))]$

- Non standard RH implementation
- Lack of reactivity
- Potential feasibility problems
  - In presence of disturbances
  - Under truncated optimization
Consider nonlinear systems

\[ \dot{x} = f(x, u) \ ; \ x \in \mathbb{R}^n \ ; \ u \in \mathbb{R}^m \ ; \ f \text{ continuous} \]

satisfying the following assumption

**Infinitely fast state excursions need infinite control**
Consider nonlinear systems

\[ \dot{x} = f(x, u) \quad ; \quad x \in \mathbb{R}^n \quad ; \quad u \in \mathbb{R}^m \quad ; \quad f \text{ continuous} \]

satisfying the following assumption

**Infinitely fast state excursions need infinite control**

For all finite horizon \( T > 0 \),

\[ \lim_{\|x_0\| \to \infty} \left[ \min_{u \in W[0,T]} \min_{t \in [0,T]} \|F(t, x_0, u)\| \right] = \infty \]

for all compact subset \( W \subset \mathbb{R}^m \).
✓ Choose a sampling period $\tau_s$

✓ Define a $\tau_s$-piece-wise constant control profile

$$U_{pwc}(\cdot, p) ; \quad p \in \mathbb{P}$$

✓ The parametrization is called "translatable" if for all $p \in \mathbb{P}$, there is $p^+ \in \mathbb{P}$ s.t.

$$u^i(p^+) = u^{i+1}(p)$$

$\forall i \in \{1, \ldots, N - 1\}$
✓ Choose a sampling period $\tau_s$

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The parametrization is called "translatable" if for all $p \in \mathbb{P}$, there is $p^+ \in \mathbb{P}$ s.t.

$$u^i(p^+) = u^{i+1}(p)$$

$\forall i \in \{1, \ldots, N - 1\}$

Notation $F(\cdot, x, p), V(\cdot, x, p)$

$$p^+ = \begin{pmatrix} e^{-\tau_s} & 0 \\ 0 & e^{-2\tau_s} \end{pmatrix} p$$
The strong contraction property

\[ \exists \gamma \in ]0, 1[ \text{ s.t. for all } x, \text{ there exists } p^c(x) \in \mathcal{P} \text{ such that} \]

\[ \min_{q \in \{1, \ldots, N\}} V(q_{\tau_s}, x, p^c(x)) \leq \gamma V(x) \]
The strong contraction property

1. \( \exists \gamma \in ]0, 1[ \text{ s.t. for all } x, \text{ there exists } p^c(x) \in \mathbb{P} \text{ such that} \)

\[
\min_{q \in \{1, \ldots, N\}} V(q \tau_s, x, p^c(x)) \leq \gamma V(x)
\]

2. \( p_c(\cdot) \) is bounded over bounded sets
The strong contraction property

1. \( \exists \gamma \in ]0, 1[ \) s.t. for all \( x \), there exists \( p^c(x) \in P \) such that

\[
\min_{q \in \{1, \ldots, N\}} V(q_{\tau_s}, x, p^c(x)) \leq \gamma V(x)
\]

2. \( p_c(\cdot) \) is bounded over bounded sets

3. \( \exists \) a continuous function \( \varphi : \mathbb{R}^n \to \mathbb{R}_+ \) s.t. for all \( x \) :

\[
\|V_{1 \to N}(\cdot, x, p^c(x))\|_\infty \leq \varphi(x) \cdot V(x)
\]

where

\[
\|V_{1 \to q}(\cdot, x, p)\|_\infty = \max_{i \in \{1, \ldots, q\}} V(i_{\tau_s}, x, p)
\]
Nonlinear Model Predictive Control
A new contractive scheme
The new contractive RH formulation
The receding-horizon state feedback

\[ u(k\tau_s + \tau) = u_1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s] \]

The open-loop optimal control problem

\[
\min_{(q, p) \in \{1, \ldots, N\} \times \mathbb{P}} X_V(q_{\tau_s}, x, p) + \alpha q_N \cdot \min \left\{ \varepsilon^2, \| V_{1\rightarrow q}(\cdot, x, p) \|_{\infty} \right\}
\]

The new contractive RH formulation

\[
V(x(t)) \quad \gamma V(x(t))
\]

\[ t \quad t + T \]
The open-loop optimal control problem

\[
\min_{(q,p) \in \{1, \ldots, N\} \times P} X V(q_{\tau s}, x, p) + \alpha q_N \cdot \min_{\{\varepsilon^2, \|V_{1 \rightarrow q}(\cdot, x, p)\|_{\infty}\}}
\]

The receding-horizon state feedback

\[
u(k_{\tau s} + \tau) = u_1(\hat{p}(x(k_{\tau s}))) \quad \forall \tau \in [0, \tau_s[\]

The open-loop optimal control problem

\[
\min_{(q,p) \in \{1,\ldots,N\} \times P_X^V(q_{\tau s}, x, p)} + \alpha_{q_{N}} \cdot \min_{\varepsilon_2, \|V_{1 \to q}(\cdot, x, p)\|_{\infty}}
\]

The receding-horizon state feedback

\[
u_k(t_{\tau s} + \tau) = u_1(\hat{p}(x_k)) \quad \forall \tau \in [0, \tau_s]
\]
The open-loop optimal control problem

\[
\min_{(q,p) \in \{1, \ldots, N\} \times \mathcal{P}} X(q_{\tau_s}, x, p) + \alpha q_N \cdot \min_{\epsilon^2, \|V_{1 \rightarrow q}(\cdot, x, p)\|_{\infty}}
\]

The receding-horizon state feedback

\[
u(k\tau_s + \tau) = u_1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s]\]

![Diagram showing the relationship between V(x(t+)) and γV(x(t+))](image-url)
The open-loop optimal control problem

\[
\min_{(q,p) \in \{1, \ldots, N\} \times \mathbb{P}_X} \quad V(q\tau_s, x, p) + \alpha \frac{q}{N} \cdot \min \left\{ \varepsilon^2, \|V_1 \rightarrow_q (\cdot, x, p)\|_{\infty} \right\}
\]
The open-loop optimal control problem

\[
\min_{(q,p) \in \{1,\ldots,N\} \times \mathbb{P}_X} \quad V(q\tau_s, x, p) + \alpha \frac{q}{N} \cdot \min \left\{ \varepsilon^2, \| V_{1\rightarrow q}(\cdot, x, p) \|_\infty \right\}
\]

The receding-horizon state feedback

\[
u(k\tau_s + \tau) = u^1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s[\]

A new contractive scheme

The new contractive RH formulation

\[
\min_{(q,p)\in\{1,\ldots,N\}\times\mathbb{P}_X} V(q\tau_r, x, p) + \alpha \frac{q}{N} \cdot \min\{\varepsilon^2, \|V_{1\rightarrow q}(\cdot, x, p)\|_\infty\}
\]

\[
\|V_{1\rightarrow q}(\cdot, x, p)\|_\infty
\]

\[
V(\cdot, x, p)
\]
The new contractive RH formulation

\[
\min_{(q,p) \in \{1, \ldots, N\} \times \mathbb{P}_X} \min \left\{ \frac{\alpha q}{N} \cdot \min \left\{ \varepsilon^2, \| V_{1-q}(\cdot, x, p) \|_\infty \right\}, \| V_{1-q}(\cdot, x, p) \|_\infty \right\}
\]

\( V(x(t_k)) \)

\( V(q \tau_s, x, p) \)

\( (\hat{q}(t_k), \hat{p}(t_k)) \) optimal solution

\( N \) Time
The new contractive RH formulation

\[
\min_{(q,p) \in \{1, \ldots, N\} \times \mathbb{P}_X} V(q \tau_s, x, p) + \alpha \frac{q}{N} \cdot \min \{ \varepsilon^2, \| V_{1 \rightarrow q} (\cdot, x, p) \|_\infty \}
\]

\[V(x(t_{k+1}))\]

\[V(x(t_k))\]

\[V(q \tau_s, x, p)\]

\[t_k \quad t_{k+1} \]

\[\hat{q}(t_k)\]

\[N \quad \text{Time}\]
The new contractive RH formulation

\[ V(q\tau_s, x, p) + \alpha \frac{q}{N} \cdot \min\{\varepsilon^2, \|V_{1\rightarrow q}(\cdot, x, p)\|_{\infty}\} \]

\[ \|V_{1\rightarrow q}(\cdot, x, p)\|_{\infty} \]

if \( \hat{q}(t_k) \geq 1 \) then

\[ \hat{q}(t_{k+1}) = \hat{q}(t_k) - 1 \]

\[ \tilde{p}(t_{k+1}) = \tilde{p}^+(t_k) \]
The new contractive RH formulation

\[ V(q \tau_s, x, p) + \alpha \frac{q}{N} \cdot \min \{ \varepsilon^2, \| V_{1 \rightarrow q}(\cdot, x, p) \|_\infty \} \]

\[ \| V_{1 \rightarrow q}(\cdot, x, p) \|_\infty \]

If \( \hat{q}(t_k) \geq 1 \) then

\[ \tilde{q}(t_{k+1}) = \hat{q}(t_k) - 1 \]

\[ \tilde{p}(t_{k+1}) = \hat{p}^+(t_k) \]

\[ V(x(t_{k+1})) \leq V(x(t_k)) - \frac{\alpha}{N} \cdot \min \{ \varepsilon^2, \| V_{1 \rightarrow \hat{q}(t_k)}(\cdot, x(t_k), \tilde{p}(t_k)) \|_\infty \} \]
Basic Result

If the following conditions hold

1. Continuity (system/parametrization)
2. Infinitely fast excursions need infinite controls
3. The control parametrization is translatable on

\[ P_X := P \cap B\left(0, \sup_{x \in \bar{B}(0, \rho(X))} \| p^c(x) \| + \varepsilon_0 \right) \subseteq P \subseteq \mathbb{R}^{np} \]

Then, \( \exists \) sufficiently small \( \varepsilon > 0 \) and \( \alpha > 0 \) such that the RH feedback is well defined and makes the origin \( x = 0 \) asymptotically stable for the resulting CL dynamics with a region of attraction that contains \( X \).
The system equations

\[
\begin{pmatrix}
  mL^2 + I & mL \cos \theta \\
  mL \cos \theta & m + M
\end{pmatrix}
\begin{pmatrix}
  \ddot{\theta} \\
  \ddot{r}
\end{pmatrix}
= \begin{pmatrix}
  mLg \sin \theta - k_\theta \dot{\theta} \\
  F + mL\dot{\theta}^2 \sin \theta - k_x \dot{r}
\end{pmatrix}
\]

A pre-compensator

\[
F = -K_{pre} \begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u
\]
The simple inverted pendulum: A self contained RH control

The system equations

\[
\begin{align*}
\dot{x}_1 &= x_3 ; \quad \dot{x}_2 = x_4 \\
\begin{pmatrix}
\dot{x}_3 \\
\dot{x}_4
\end{pmatrix} &= [M(x)]^{-1} \begin{pmatrix}
mLg \sin(x_1) - k_{\theta} \cdot x_3 \\
-K_{pre_1} x_2 - K_{pre_2} x_4 + mL x_3^2 \sin(x_1) - k_x x_4 + u
\end{pmatrix}
\end{align*}
\]
Control parametrization

\[ u^i(p) = p \cdot e^{-t_i/t_r} \; ; \; \; t_i = \frac{(i - 1)\tau_s}{N} \]

where \( p \in \mathbb{P}(x) := [p_{min}(x), p_{max}(x)] \) s.t

\[
\begin{align*}
p_{min}(x) &= -F_{max} + K_{pre_1}x_2 + K_{pre_2}x_4 \\
p_{max}(x) &= +F_{max} + K_{pre_1}x_2 + K_{pre_2}x_4
\end{align*}
\]
Use the contractive RH formulation given by:

\[
V(x) = \frac{1}{2} \left[ \dot{\theta}^2 + \beta r^2 + \dot{r}^2 \right] + \left[1 - \cos(\theta)\right]^2
\]

\[
\min_{(q,p)\in\{1,\ldots,N\} \times \mathbb{P}(x)} \left\{ V(q\tau_s, x, p) + \frac{\alpha}{N} \cdot \min\{\varepsilon, \|V_{1\rightarrow q}(\cdot, x, p)\|_\infty\} \right\}
\]

\[
u(k\tau_s + \tau) = u^1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s[\]
The parameters of the controller

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>signification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_s$</td>
<td>0.4 s</td>
<td>sampling period</td>
</tr>
<tr>
<td>$N$</td>
<td>8</td>
<td>horizon length</td>
</tr>
<tr>
<td>$t_r$</td>
<td>0.2</td>
<td>Constant for the control param.</td>
</tr>
<tr>
<td>$\alpha = \varepsilon$</td>
<td>0.01</td>
<td>cost function parameters</td>
</tr>
<tr>
<td>$K_{pre}$</td>
<td>(2.5, 10.0)</td>
<td>Pre-compensation gain</td>
</tr>
<tr>
<td>$F_{max}$</td>
<td>$\in {1.0, 2.0}$</td>
<td>saturation level on $F$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\in 10$</td>
<td>weighting coefficient on $r$</td>
</tr>
</tbody>
</table>

Runs on a 1.3 GHz Pentium-III
Nonlinear Model Predictive Control

Illustrative examples

The simple inverted pendulum: A self contained RH control

\[ \theta \ (\text{deg}) \]

\[ r \ (\text{m}) \]

\[ F(\text{N}) \]

Computation time (\text{ms})

Time (s)

Sampling periods
The double inverted pendulum: a hybrid control scheme

\[ L_1 = 2l_1 \]
\[ J_1 = \frac{1}{2} m_1 l_1^2 \]
\[ J_2 = \frac{1}{2} m_2 l_2^2 \]
\[ h_1 = m + m_1 + m_2 \]
\[ h_2 = m_1 l_1 + m_2 l_2 \]
\[ h_3 = m_2 l_2 \]
\[ h_4 = m_1 l_1^2 + m_2 L_1^2 + J_1 \]
\[ h_5 = m_2 l_2 L_1 \]
\[ h_6 = m_2 l_2^2 + J_2 \]
\[ h_7 = (m_1 l_1 + m_2 L_1)g \]
\[ h_8 = m_2 l_2 g \]
The double inverted pendulum: a hybrid control scheme

**System equations**

\[ h_1 \ddot{r} + h_2 \ddot{\theta}_1 \cos \theta_1 + h_3 \ddot{\theta}_2 \cos \theta_2 = h_2 \dot{\theta}_1^2 \sin \theta_1 + h_3 \dot{\theta}_2^2 \sin \theta_2 + F \]

\[ h_2 \ddot{r} \cos \theta_1 + h_4 \ddot{\theta}_1 + h_5 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) = h_7 \sin \theta_1 - h_5 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \]

\[ h_3 \ddot{r} \cos \theta_2 + h_5 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + h_6 \ddot{\theta}_2 = h_5 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + h_8 \sin \theta_2 \]

\[
L_1 = 2l_1 \\
J_1 = \frac{1}{2} m_1 l_1^2 \\
J_2 = \frac{1}{2} m_2 l_2^2 \\
h_1 = m + m_1 + m_2 \\
h_2 = m_1 l_1 + m_2 l_2 \\
h_3 = m_2 l_2 \\
h_4 = m_1 l_1^2 + m_2 L_1^2 + J_1 \\
h_5 = m_2 L_1 \\
h_6 = m_2 l_2^2 + J_2 \\
h_7 = (m_1 l_1 + m_2 L_1)g \\
h_8 = m_2 l_2 g
\]
Pre-compensation

\[ F = -K_{pre} \cdot \left( \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} \right) + u \]
Control parametrization

\[ u^i(p) = p_1 \cdot e^{\lambda_1 t_i} + p_2 e^{-\lambda_2 t_i} \quad ; \quad t_i = \frac{(i - 1) \tau_s}{N} \]

\[ p_{\min}(x) := \frac{1}{2} \left[ -F_{\max} + K_{pre} \left( \frac{r}{\dot{r}} \right) \right] \quad ; \quad p_{\max}(x) := \frac{1}{2} \left[ +F_{\max} + K_{pre} \left( \frac{r}{\dot{r}} \right) \right] \]
The contractive RH controller

\[ V(x) = \frac{h_4}{2} \dot{\theta}_1^2 + \frac{h_6}{2} \dot{\theta}_2^2 + h_5 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + h_7 \left[ 1 - \cos(\theta_1) \right] + \]
\[ + h_8 \left[ 1 - \cos(\theta_2) \right] + h_1 \left[ r^2 + \dot{r}^2 \right] \]

\[ u(k\tau_s + t) = K_{RH}(x(k\tau_s)) := u^1(\hat{p}(x(k\tau_s))) ; \quad t \in [0, \tau_s] \]
The contractive RH controller

\[
V(x) = \frac{h_4}{2} \dot{\theta}_1^2 + \frac{h_6}{2} \dot{\theta}_2^2 + h_5 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + h_7 \left[ 1 - \cos(\theta_1) \right] + \\
+ h_8 \left[ 1 - \cos(\theta_2) \right] + h_1 \left[ r^2 + \dot{r}^2 \right]
\]

\[
u(k\tau_s + t) = K_{RH}(x(k\tau_s)) := u^1(\hat{p}(x(k\tau_s))) ; \quad t \in [0, \tau_s[\]

A local LQR controller

\[
K_L(x) = -L \cdot \begin{pmatrix}
x_1^m \\
x_2^m \\
x_3 \\
\vdots \\
x_6
\end{pmatrix}
\]

solving the discrete time Riccati equation

\[
A_d^T S A_d - S - (A_d^T S B_d)(R + B_d^T S B_d)(B_d^T S A_d) + Q = 0
\]
To summarize, the hybrid controller is given by:

\[
    u(kT_s + \tau) = \begin{cases} 
        K_{RH}(x(kT_s)) & \text{if } \|x(kT_s)\|_S^2 > \eta \\
        K_L(x(kT_s)) & \text{otherwise}
    \end{cases}
\]
To summarize, the hybrid controller is given by

\[ u(k\tau_s + \tau) = \begin{cases} 
K_{RH}(x(k\tau_s)) & \text{if } \|x(k\tau_s)\|_S^2 > \eta \\
K_L(x(k\tau_s)) & \text{otherwise}
\end{cases} \]

### The parameters of the controller

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<td>(\tau_s)</td>
<td>0.3 s</td>
<td>sampling period</td>
</tr>
<tr>
<td>(N)</td>
<td>10</td>
<td>horizon length</td>
</tr>
<tr>
<td>(L)</td>
<td>(360, 29)</td>
<td>(linear controller gain)</td>
</tr>
<tr>
<td>((\lambda_1, \lambda_2))</td>
<td>(100, 20)</td>
<td>Control parametrization</td>
</tr>
<tr>
<td>(\eta)</td>
<td>1.0</td>
<td>switching threshold</td>
</tr>
<tr>
<td>(i_{max})</td>
<td>20</td>
<td>Max number of function evaluation</td>
</tr>
</tbody>
</table>
Nonlinear Model Predictive Control
Illustrative examples
The double inverted pendulum: a hybrid control scheme

\( \theta_1(\text{deg}) \)
\( \theta_2(\text{deg}) \)
\( r(\text{m}) \)

Time (s)
F(N)
Optimal cost \( \hat{J}(x) \)
comp. times (ms)

Sampling period

\( \theta_1(\text{deg}) \)
\( \theta_2(\text{deg}) \)
\( r(\text{m}) \)

Time (s)

M. Alamir (–) Nonlinear Model Predictive Control 8,15 Novembre 2005 71 / 76
Double inverted pendulum

le film
The twin pendulum
The twin pendulum

- \( \theta_1/(2\pi) \) et \( \theta_2/(2\pi) \)
- Position chariot \( r \) (m)
- La fonction E
- Accélération du chariot \( d^2r/dt^2 \) (m/s²)
Nonlinear constrained NMPC for maximizing the production in polymerization processes.
Further readings

N. Marchand and M. Alamir

*Numerical stabilization of a rigid spacecraft with two actuators*

*Journal of dynamic systems, measurements and control.*
Further readings

M. Alamir and F. Boyer

*Fast generation of attractive trajectories for an under-actuated satellite: Application to feedback control design*

Further readings

M. Alamir

*Nonlinear Receding Horizon sub-optimal guidance law for minimum interception time problem*

Further readings

M. Alamir and N. Marchand

*Constrained Minimum Time Oriented Feedback Control For the Stabilization of Nonholonomic Systems in Chained Form*

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A. Hably, N. Marchand and M. Alamir

Constrained Minimum-Time Oriented Stabilization of Extended Chained Form Systems

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M. alamir and H. Khennouf

*Discontinuous Receding Horizon Control Based Stabilizing Feedback for Nonholonomic Systems in Power Form*

**CDC. New Orleans, (1995).**
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A. Chemori and M. Alamir

Limit Cycle Generation for a Class of Nonlinear Systems with jumps using a Low Dimensional Predictive Control

Further readings

A. Chemori and M. Alamir

Multi-step Limit Cycle Generation for Rabbit’s Walking Based on a Nonlinear Low dimensional Predictive Control Scheme

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M. Alamir, F. Ibrahim and J. P. Corriou

*A Flexible Nonlinear Model Predictive Control Scheme for Quality/Performance Handling in Nonlinear SMB Chromatography*

Further readings

S. A. Attia, M. Alamir and C. Canudas de Wit

*A Voltage Collapse Avoidance in* Power Systems *: A Receding Horizon Approach*

Further readings

M. Alamir and G. Bornard

*On the stability of receding horizon control of nonlinear discrete-time systems*

Further readings

M. Alamir and N. Marchand

*Numerical Stabilization of Nonlinear Systems: Exact Theory and Approximate Numerical Implementation*

Further readings

M. Alamir and G. Bornard

*Stability of truncated Infinite Constrained Receding Horizon Scheme: The General Nonlinear Case*

Further readings

M. Alamir and I. Balloul

Robust Constrained Control Algorithm for General Batch Processes

Further readings

M. Alamir

*A new Path Generation Based Receding Horizon Formulation for Constrained Stabilization of Nonlinear Systems*

Further readings

M. Alamir

*A Low Dimensional Contractive NMPC Scheme for Nonlinear Systems Stabilization: Theoretical Framework and Numerical Investigation on Relatively Fast Systems.*

**Workshop on Assessment and Future Directions of NMPC, Freudenstadt, Germany** (2005).
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