Nonlinear Model Predictive Control

More than an introduction...

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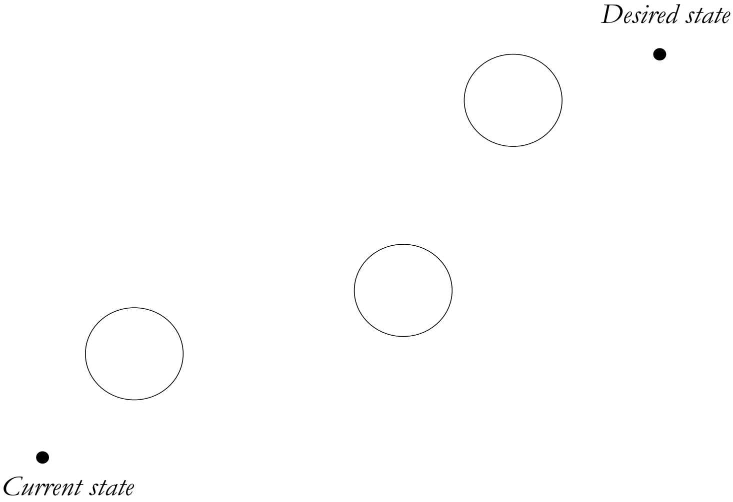
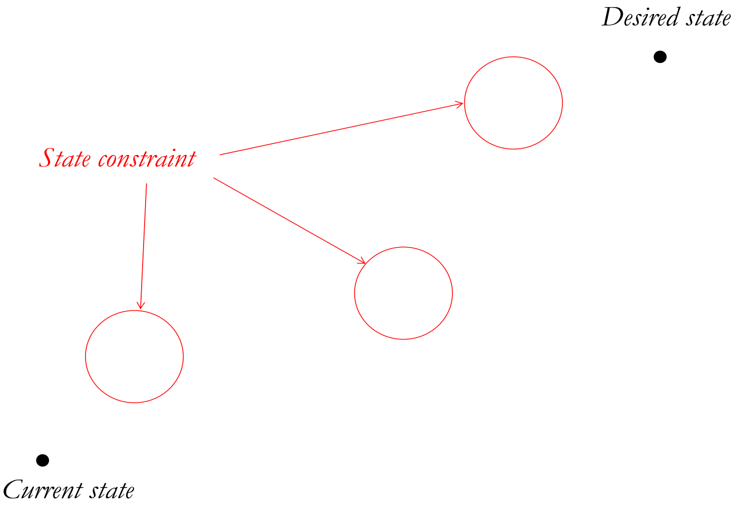
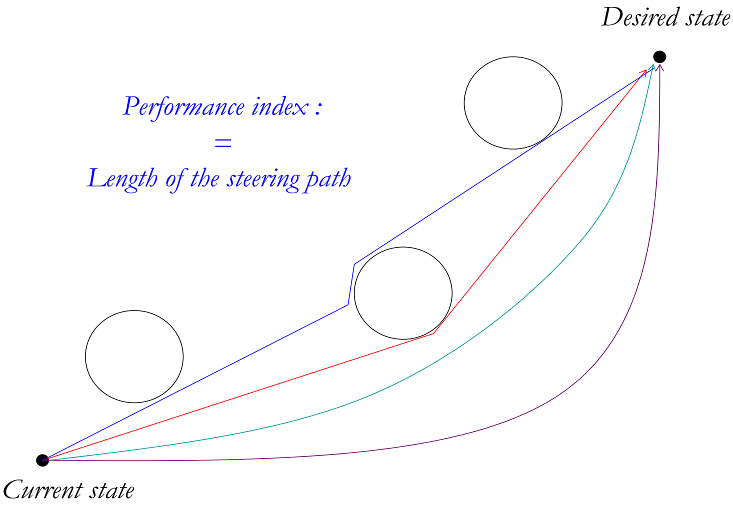
Atelier technique

Regardless of the control strategy being used, the following keywords has to be addressed :

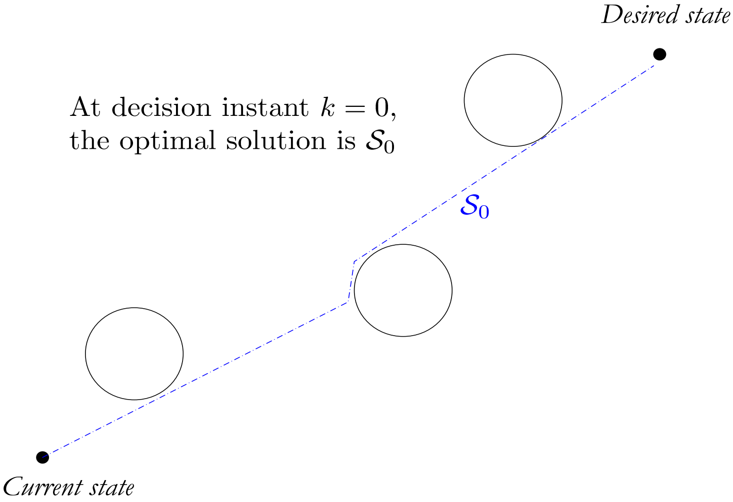
- **System model**
  (State, control, measurement, disturbance)
- **Constraints**
  (on state, control)
- **performance index**
  (Operational cost, energy consumption, tracking quality)
- **Stability**
- **Robustness**


Desired state
Current state

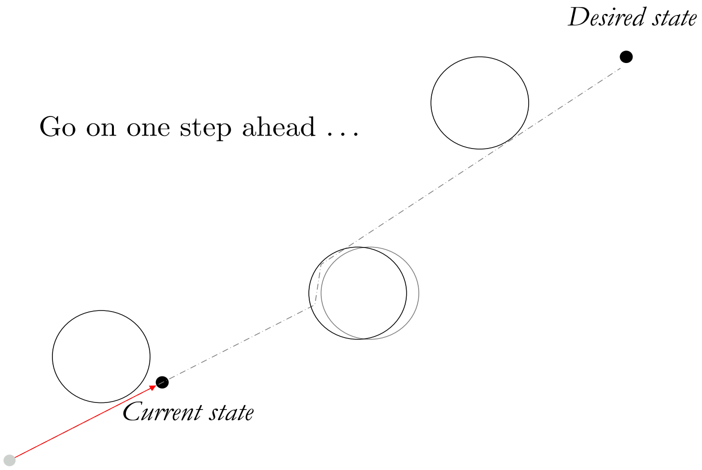

Desired state
State constraint
Current state


Desired state
Performance index :
=
Length of the steering path
Current state

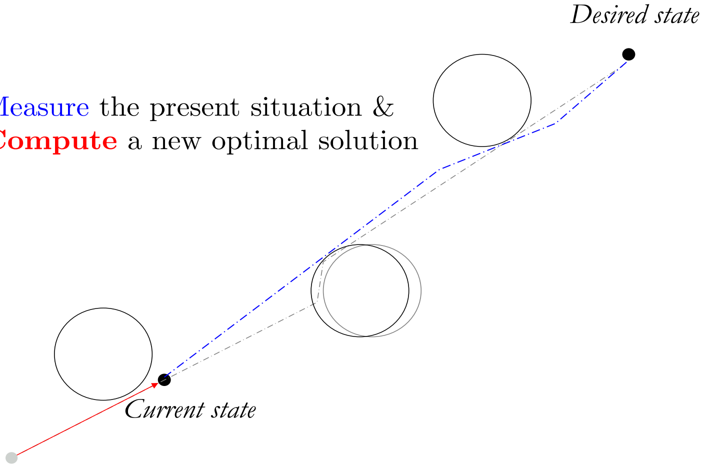
At decision instant $k = 0$,
the optimal solution is $\mathcal{S}_0$



Go on one step ahead . . .



Measure the present situation &
Compute a new optimal solution



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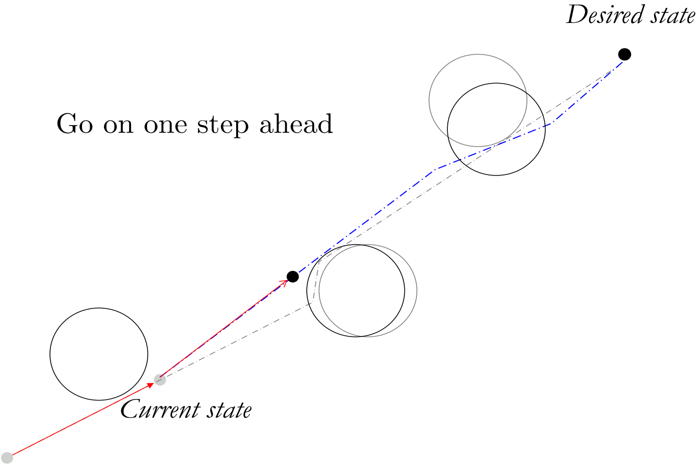

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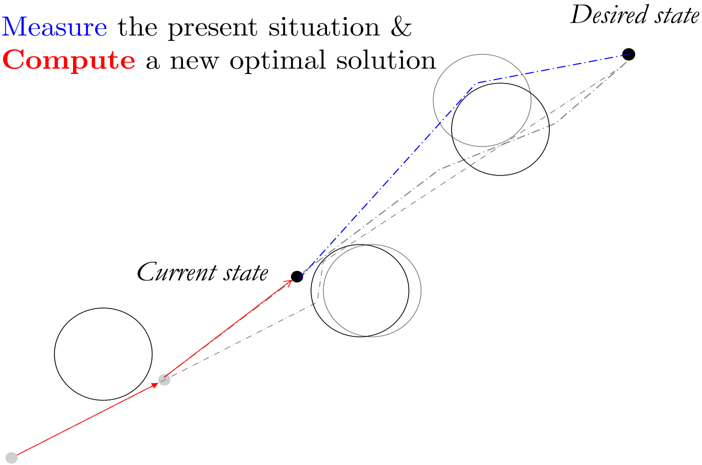
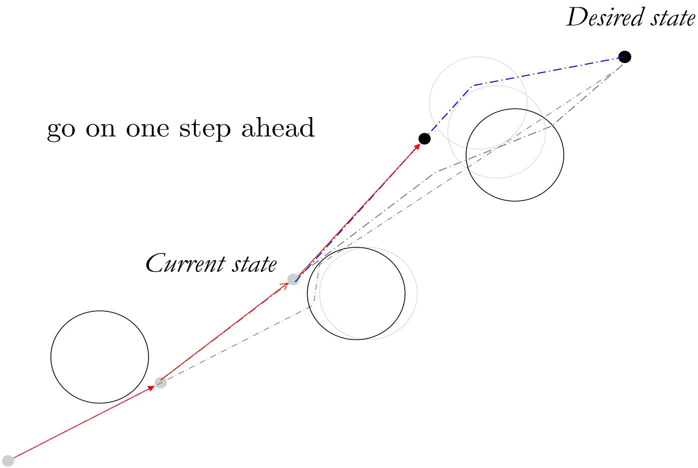
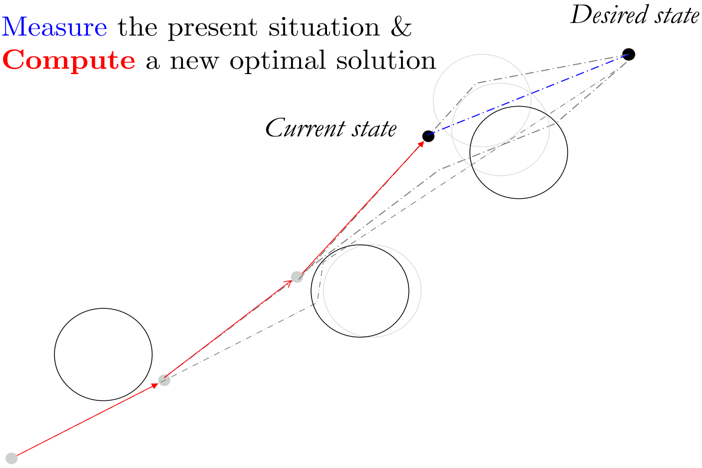
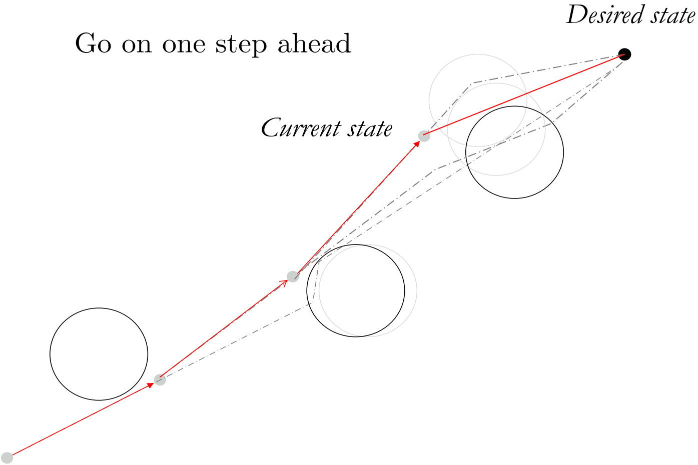


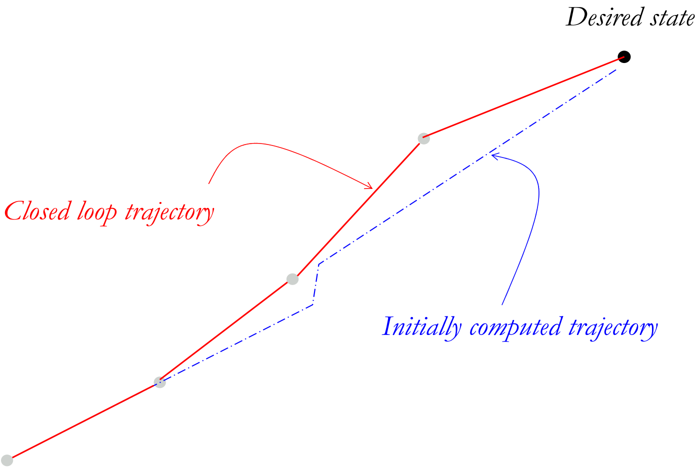

Desired state
go on one step ahead
Current state

Measure the present situation &
Compute a new optimal solution



Go on one step ahead




Desired state
Closed loop trajectory
Initially computed trajectory

A simple feedback principle (informal)

- At each decision instant, evaluate the situation

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- Keep doing

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  $\mathbf{u}^0(x(k)) := \big(u^0(k; x(k)) \quad u^0(k+1; x(k)) \quad \dots \quad u^0(k+i; x(k)) \quad \dots\big)$

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- **Apply the control** $u^0(k;x(k))$ on the sampling period $[k, k+1]$

A state feedback

We have defined a sampled state feedback

$$u(k) = u^0(k;x(k))$$

A key task . . .

- At decision instant k, measure the state $x(k)$
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We need an optimization problem

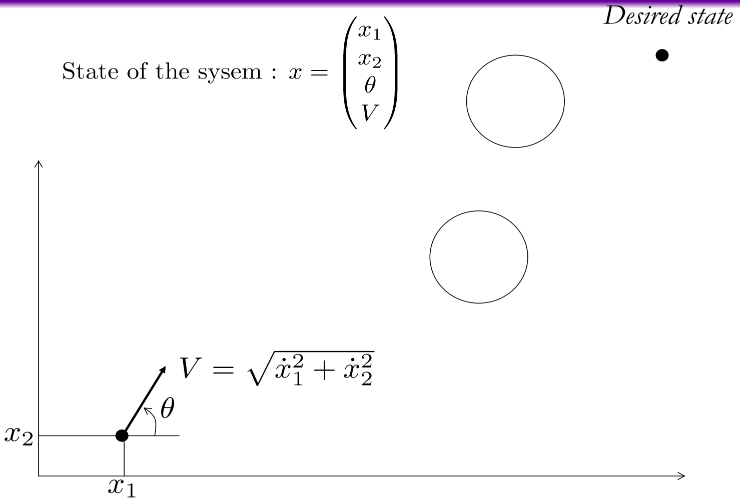
$$\mathcal{P}(x(k)) \quad : \quad \min_{\mathbf{u}} \Big\{ V(x(k), \mathbf{u}) \quad | \quad \mathbf{u} \in \mathcal{U}(x(k)) \Big\}$$

$$\mathbf{u}^0(x(k)) \qquad \text{is A solution of } \mathcal{P}(x(k))$$

How to define & Solve $\mathcal{P}(x)$ for our example ?

Step 1 : Write down the system model

State of the sysem : $x = \begin{pmatrix} x_1 \\ x_2 \\ \theta \\ V \end{pmatrix}$



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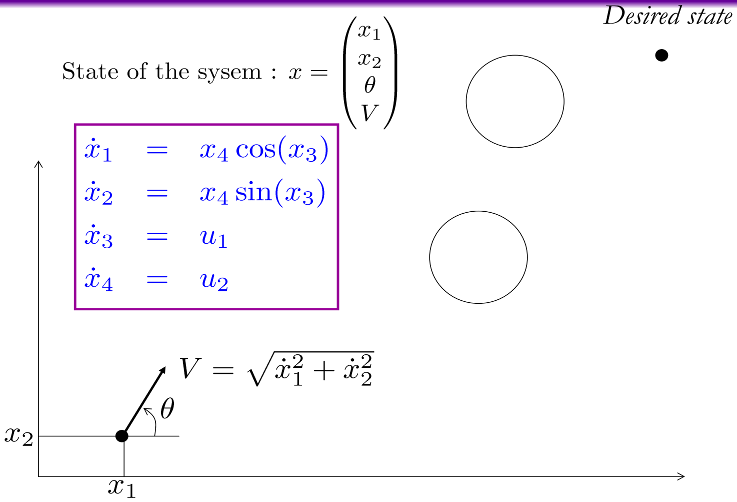
$$\begin{aligned} \dot{x}_1 &= x_4 \cos(x_3) \\ \dot{x}_2 &= x_4 \sin(x_3) \\ \dot{x}_3 &= u_1 \\ \dot{x}_4 &= u_2 \end{aligned}$$



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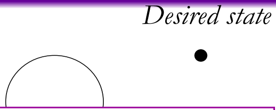

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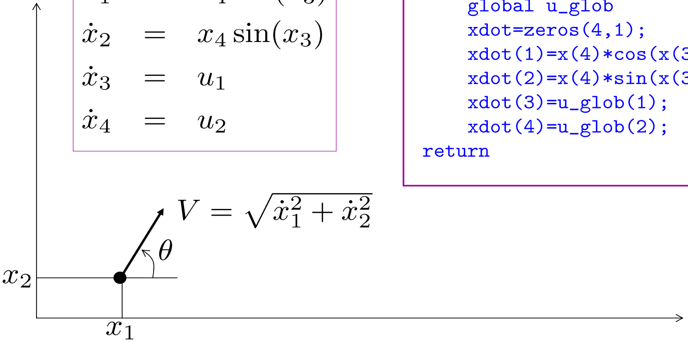
```
function xdot=car(t,x)
    global u_glob
    xdot=zeros(4,1);
    xdot(1)=x(4)*cos(x(3));
    xdot(2)=x(4)*sin(x(3));
    xdot(3)=u_glob(1);
    xdot(4)=u_glob(2);
return
```



Step 2 : Obtain the sampled-time model

Given the continuous system

$$\dot{x} = f_c(x, u)$$

Compute the implicit τ-discrete dynamics

$$x^+(k) = x(k+1) = f(x(k), u(k))$$

where $x(k+1)$ is the solution at instant τ of

$$\dot{\xi} = f_c(\xi, u(k)) \quad ; \quad \xi(0) = x(k)$$

```
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    global u_glob
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```

Continuous system

```
function xplus=car_d(x,u,tau)
    global u_glob
    u_glob=u;
    [tt,xx]=ode45(@car,[0,tau],x);
    xplus=xx(end,:)';
return
```

τ-sampled system

Some notations before we continue

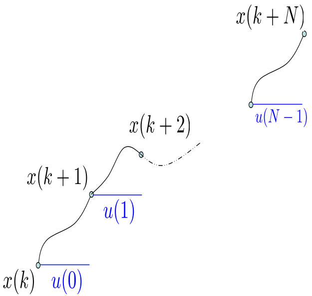
Consider

$$x^+ = f(x,u)$$

$$\mathbf{u} := (u(0), u(1), \dots, u(N-1))$$

Notation

$$x^{\mathbf{u}}(\cdot, x(k)) := \Big\{ x(k+i) \Big\}_{i=0}^{N}$$



$$x(k+i) = f\Big(x(k+i-1), u(i-1)\Big)$$

Recall

$$\mathcal{P}(x(k)) \quad : \quad \min_{\mathbf{u}} \Big\{ V(x(k), \mathbf{u}) \quad | \quad \mathbf{u} \in \mathcal{U}(x(k)) \Big\}$$

$$\mathbf{u}^0(x(k)) \qquad \text{is A solution of } \mathcal{P}(x(k))$$

Step 3 : Write down the constraints

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The final constraint

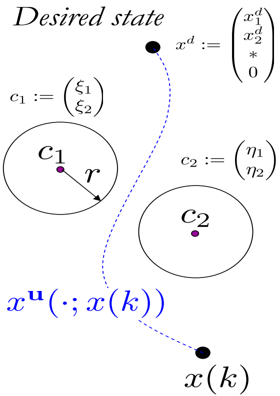
$$x^{\mathbf{u}}(N;x(k)) - x^d = 0$$

Obstacle avoidance

$$r - \min_{i\in\{1,2\}} d\Big(x^{\mathbf{u}}(\cdot;x(k)), c_i)\Big) \leq 0$$

Saturation constrainte

$$\mathbf{u} \in [u_{min}, u_{max}]^N$$



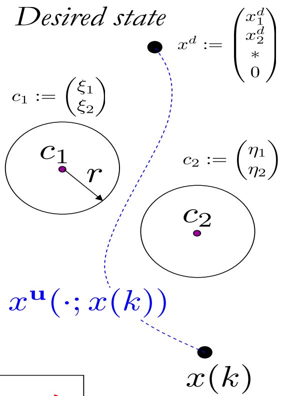
Step 3 : Write down the constraints

Equality constraints

$$G_1(\mathbf{u}, x(k)) = 0$$

Inequality constraints

$$G_2(\mathbf{u}, x(k)) \leq 0$$



$$\mathcal{U}(x(k)) := \left\{ \mathbf{u} \quad | \quad G_1 = 0 \text{ and } G_2 \leq 0 \right\}$$

Step 3 : Write down the constraints

```
function [G1,G2]=constraintes(les_u,x,tau)
    global xd xi eta u_min u_max
    N=length(les_u(:,1));
    xu=zeros(N,4); xu(1,:)=x';
    d(1)=min(norm(xu(1,1:2)-xi),norm(xu(1,1:2)-eta));
    for i=2:N,
       xu(i,:)=car_d(xu(i-1,:)',lesu(i-1,:)',tau);
       d(i)=min(norm(xu(i,1:2)-xi),norm(xu(i,1:2)-eta));
    end
    cond_min=max(max(ones(N,1)*u_min'-lesu));
    cond_max=max(max(lesu-ones(N,1)*u_max'));
    G1=xu(N,:)'-xd;
    G2=[r-min(d);cond_min;cond_max];
return
```

Recall

$$\mathcal{P}(x(k)) \quad : \quad \min_{\mathbf{u}} \Big\{ V(x(k),\mathbf{u}) \quad | \quad \mathbf{u} \in \mathcal{U}(x(k)) \Big\}$$

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Write down the performance index

$$V(x(k), \mathbf{u}) := \sum_{i=1}^{N} \left\| x^{\mathbf{u}}(i; x(k)) - x^d \right\|^2$$

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$$V(x(k),\mathbf{u}) := \sum_{i=1}^{N} \left\| x^{\mathbf{u}}(i; x(k)) - x^d \right\|^2$$

```
function f=cost(lesu,x,tau)
    global xd
    N=length(les_u(:,1));
    xu=zeros(N,4);
    xu(1,:)=x';
    f = 0;
    for i=2:N,
       xu(i,:)=car_d(xu(i-1,:)',lesu(i-1,:)',tau);
       f = f+norm(xu(i,:)'-xd)^2;
    end
return
```

```
FMINCON finds a constrained minimum of a function of several variables.
    FMINCON attempts to solve problems of the form:
        min F(X)  subject to:  A*X  <= B, Aeq*X  = Beq (linear constraints)
         X                       C(X) <= 0, Ceq(X) = 0   (nonlinear constraints)
                                 LB <= X <= UB

    X=FMINCON(FUN,X0,A,B) starts at X0 and finds a minimum X to the function
    FUN, subject to the linear inequalities A*X <= B. FUN accepts input X and
    returns a scalar function value F evaluated at X. X0 may be a scalar,
    vector, or matrix.

    X=FMINCON(FUN,X0,A,B,Aeq,Beq) minimizes FUN subject to the linear equalities
    Aeq*X = Beq as well as A*X <= B. (Set A=[] and B=[] if no inequalities exist.)

    X=FMINCON(FUN,X0,A,B,Aeq,Beq,LB,UB) defines a set of lower and upper
    bounds on the design variables, X, so that a solution is found in
    the range LB <= X <= UB. Use empty matrices for LB and UB
    if no bounds exist. Set LB(i) = -Inf if X(i) is unbounded below;
    set UB(i) = Inf if X(i) is unbounded above.

    X=FMINCON(FUN,X0,A,B,Aeq,Beq,LB,UB,NONLCON) subjects the minimization to the
    constraints defined in NONLCON. The function NONLCON accepts X and returns
    the vectors C and Ceq, representing the nonlinear inequalities and equalities
    respectively. FMINCON minimizes FUN such that C(X)<=0 and Ceq(X)=0.
    (Set LB=[] and/or UB=[] if no bounds exist.)
```

Coming next ...

- Existence of solutions
- Closed-loop stability

Existence of solutions

$$\mathcal{P}(x) \ : \ \min_{\mathbf{u}} \left\{ V(x,\mathbf{u}) \quad | \quad \mathbf{u} \in \mathcal{U}(x) \right\}$$

$$\mathcal{U}(x) := \left\{ \mathbf{u} \in \mathbb{U}^N \quad | \ \forall i \quad x^{\mathbf{u}}(i;x) \in \mathbb{X} \quad \text{and} \quad x^{\mathbf{u}}(N,x) \in X_f \right\}$$

in which $\mathbb{U}$ compact ; $\mathbb{X}$ compact ; $X_f = \{x^d\} \subset \mathbb{X}$ closed.

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No global definition under bounded control

one must have $x \in X_N$ the subset of states from which X_f is accessible by N-step controls in $\mathbb{U}$ and trajectories in $\mathbb{X}$.

Existence of solutions

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in which $\mathbb{U}$ compact ; $\mathbb{X}$ compact ; $X_f = \{x^d\} \subset \mathbb{X}$ closed.

for all $x \in X_N$, main arguments

$V(x,\cdot)$ is continuous and $\mathcal{U}(x)$ is compact.

Close-loop stability

$$\mathcal{P}(x)\ :\ \min_{\mathbf{u}}\left\{V(x,\mathbf{u})\quad|\quad \mathbf{u}\in\mathcal{U}(x)\right\}$$

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$$V(x,\mathbf{u}) = F(x(N;x)) + \sum_{i=0}^{N} L(x^{\mathbf{u}}(i;x),u(i))$$

$$x^d = 0\quad ;\quad X_f=\{0\}\quad ;\quad f(0,0)=0$$

Close-loop stability

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$$x^d = 0 \quad ; \quad X_f = \{0\} \quad ; \quad f(0,0) = 0$$

An optimal solution :

$$\mathbf{u}^0(x) := \left(u^0(0;x) \quad u^0(1;x) \quad \dots \quad u^0(N-1;x) \right) \in \mathbb{U}^N$$

Close-loop stability

$$\mathcal{P}(x) \;:\; \min_{\mathbf{u}}\left\{ V(x,\mathbf{u}) \quad | \quad \mathbf{u} \in \mathcal{U}(x) \right\}$$

$$\mathcal{U}(x) := \left\{ \mathbf{u} \in \mathbb{U}^N \mid \forall i \quad x^{\mathbf{u}}(i;x) \in \mathbb{X} \quad \text{and} \quad x^{\mathbf{u}}(N,x) \in X_f \right\}$$

$$V(x,\mathbf{u}) = F(x(N;x)) + \sum_{i=0}^{N} L(x^{\mathbf{u}}(i;x), u(i))$$

$$x^d = 0 \quad ; \quad X_f = \{0\} \quad ; \quad f(0,0) = 0$$

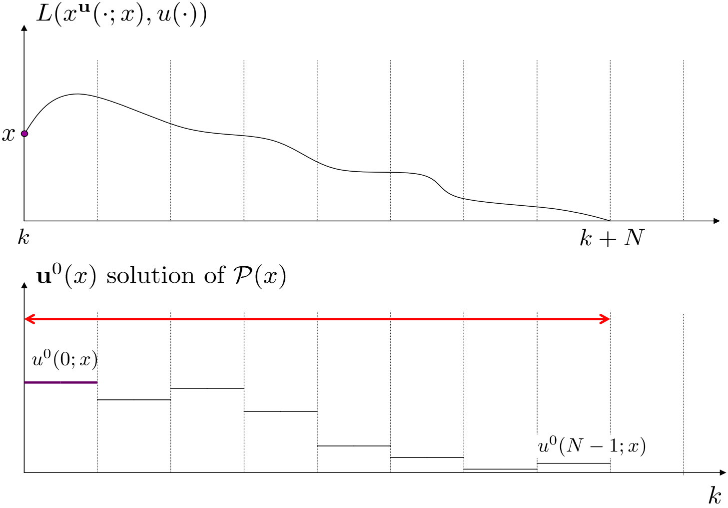
An optimal solution :

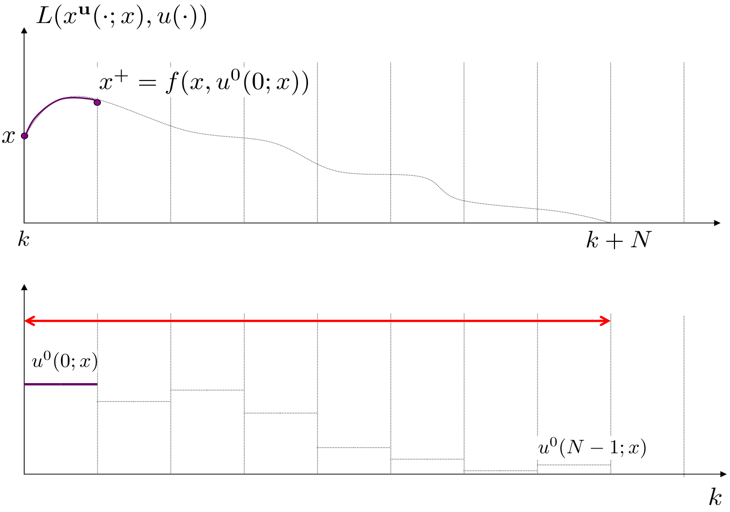
$$\mathbf{u}^0(x) := \left(u^0(0;x) \quad u^0(1;x) \quad \dots \quad u^0(N-1;x) \right) \in \mathbb{U}^N$$

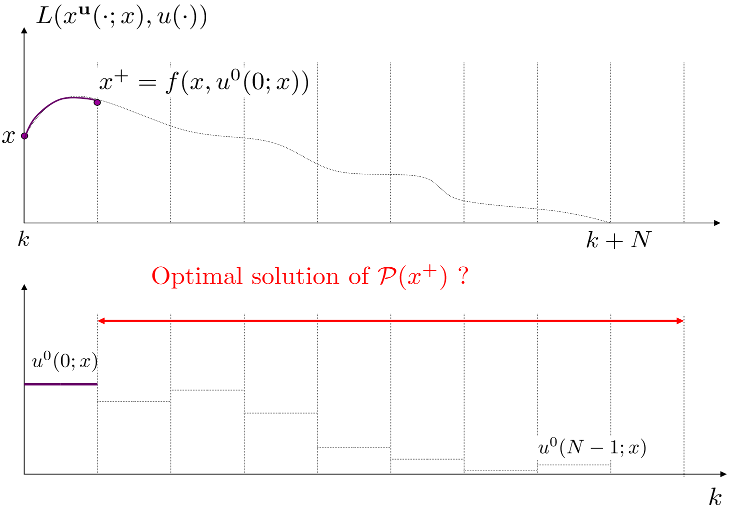
Sampled state feedback : $\kappa^0(x) := u^0(0;x)$

Study the stability of

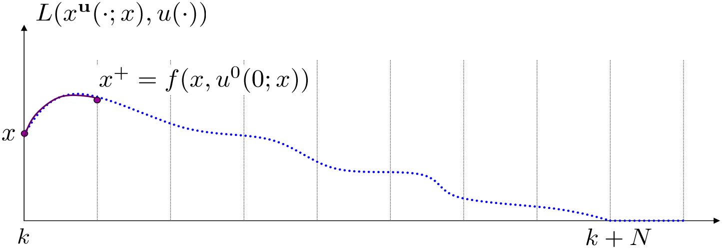
$$x^+ = f(x, u^0(0; x))$$


$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
x
k
$k+N$
$\mathbf{u}^0(x)$ solution of $\mathcal{P}(x)$
$u^0(0;x)$
$u^0(N-1;x)$
k

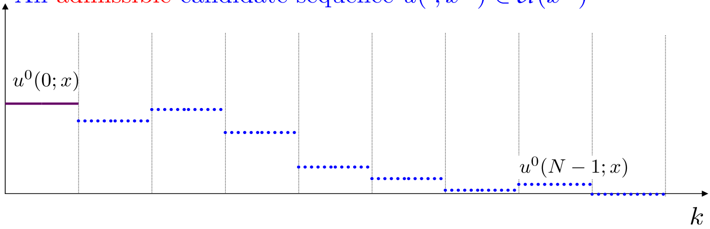

$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$x^+ = f(x, u^0(0;x))$
x
k
$k+N$
$u^0(0;x)$
$u^0(N-1;x)$
k

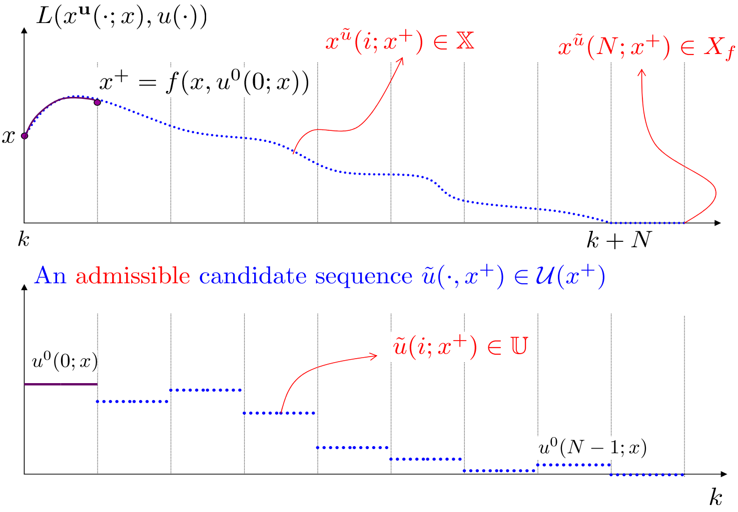


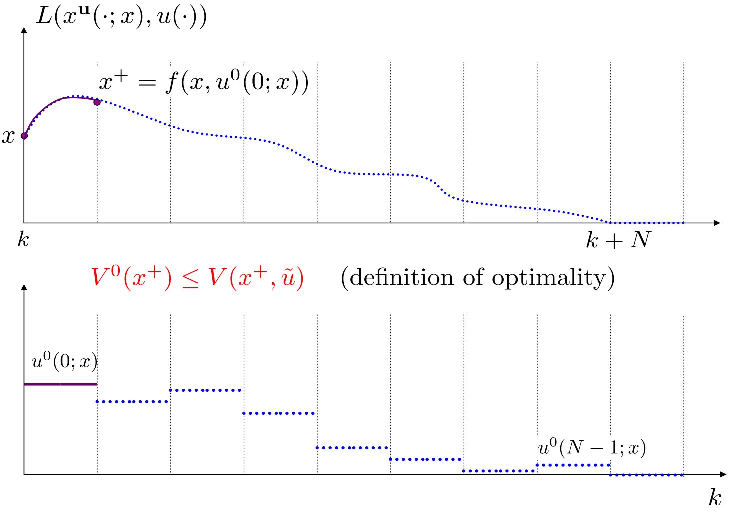
$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$x^+ = f(x,u^0(0;x))$
x
k
$k+N$
Optimal solution of $\mathcal{P}(x^+)$?
$u^0(0;x)$
$u^0(N-1;x)$
k

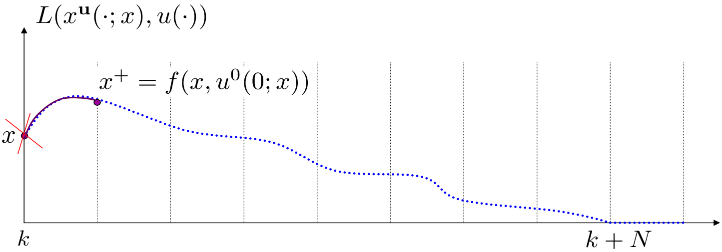


An admissible candidate sequence $\tilde{u}(\cdot, x^+) \in \mathcal{U}(x^+)$

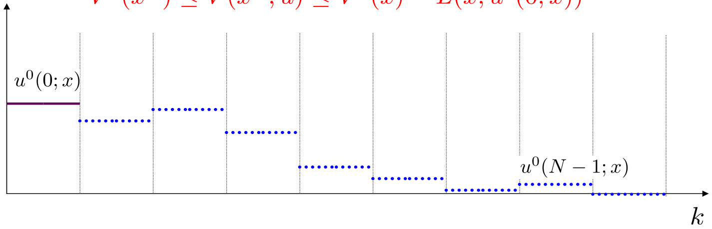



$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$x^{\tilde{u}}(i;x^+) \in \mathbb{X}$
$x^{\tilde{u}}(N;x^+) \in X_f$
$x^+ = f(x,u^0(0;x))$
x
k
$k+N$
An admissible candidate sequence $\tilde{u}(\cdot,x^+) \in \mathcal{U}(x^+)$
$\tilde{u}(i;x^+) \in \mathbb{U}$
$u^0(0;x)$
$u^0(N-1;x)$
k


$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$x^+ = f(x,u^0(0;x))$
x
k
$k+N$
$V^0(x^+) \leq V(x^+,\tilde{u})$ (definition of optimality)
$u^0(0;x)$
$u^0(N-1;x)$
k



$$V^0(x^+) \le V(x^+,\tilde{u}) \le V^0(x) - L(x,u^0(0;x))$$



Closed loop stability

Therefore, on the closed loop trajectory :

$$V^0(x(k+1)) \leq V^0(x(k)) - L\Big(x(k), \kappa^0(x(k))\Big)$$

therefore,

$$\lim_{k\to\infty} L\Big(x(k), \kappa^0(x(k))\Big) = 0$$

Consequently, if $L(\cdot,\cdot)$ is a **continuous positive definite function in** x, one has

$$\lim_{k\to\infty} x(k) = 0$$

Close-loop stability

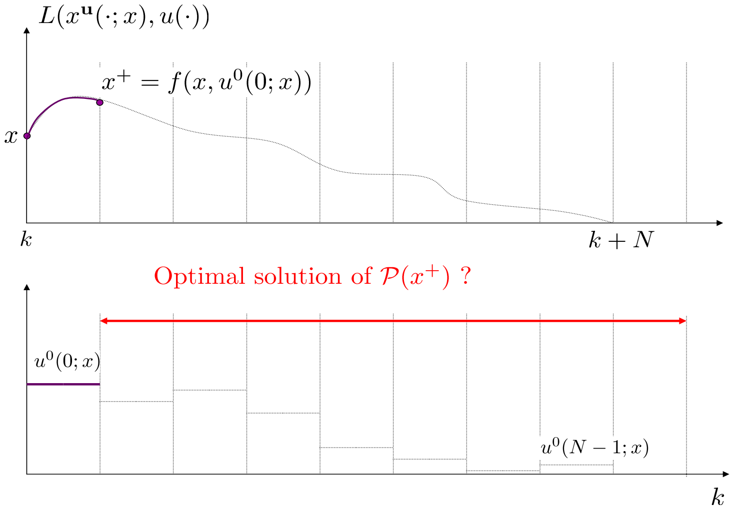
$$\mathcal{P}(x)\ :\ \min_{\mathbf{u}}\left\{V(x,\mathbf{u})\quad|\quad \mathbf{u}\in\mathcal{U}(x)\right\}$$

$$\mathcal{U}(x):=\left\{\mathbf{u}\in\mathbb{U}^N\mid\forall i\quad x^{\mathbf{u}}(i;x)\in\mathbb{X}\quad\text{and}\quad x^{\mathbf{u}}(N,x)\in X_f\right\}$$

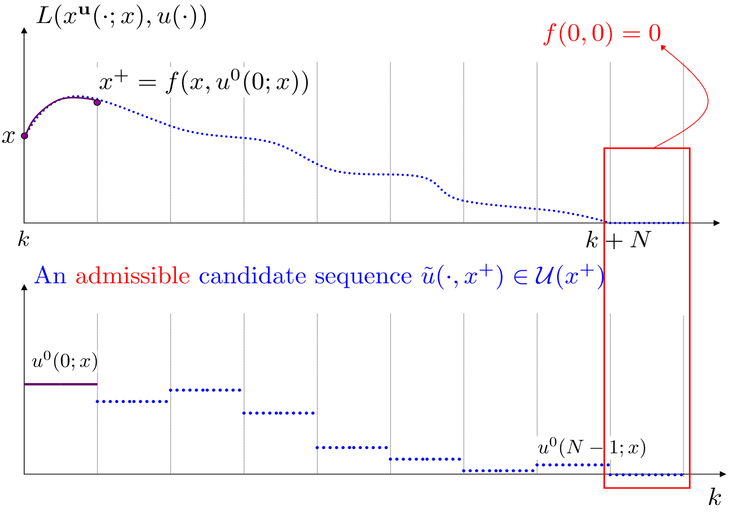
$$V(x,\mathbf{u})=F(x(N;x))+\sum_{i=0}^{N}L(x^{\mathbf{u}}(i;x),u(i))$$

$$x^d=0\quad;\quad X_f=\{0\}\quad;\quad f(0,0)=0$$

The final equality constraint : A key role



$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$x^+ = f(x,u^0(0;x))$
x
k
$k+N$
Optimal solution of $\mathcal{P}(x^+)$?
$u^0(0;x)$
$u^0(N-1;x)$
k



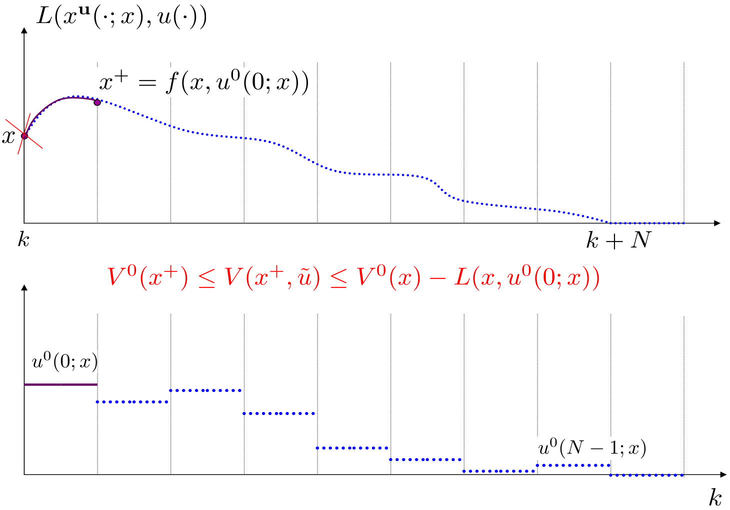
$L(x^{\mathbf{u}}(\cdot;x), u(\cdot))$
$f(0,0) = 0$
$x^+ = f(x, u^0(0;x))$
x
k
$k+N$
An admissible candidate sequence $\tilde{u}(\cdot, x^+) \in \mathcal{U}(x^+)$
$u^0(0;x)$
$u^0(N-1;x)$
k

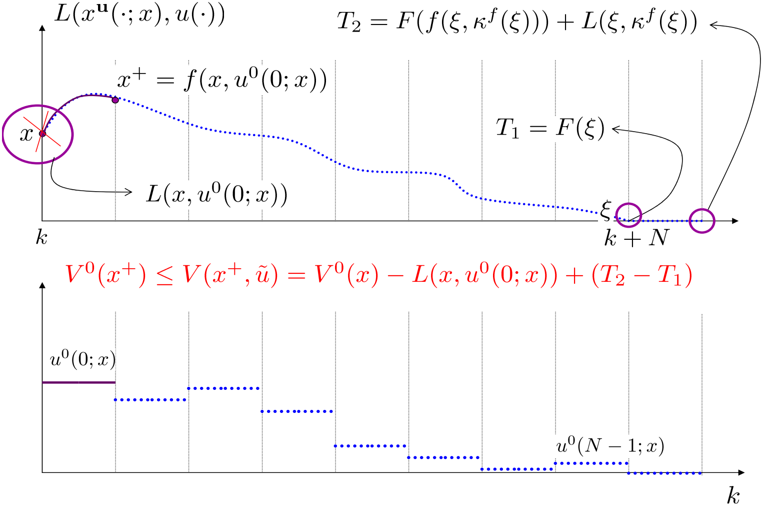
First useful property of $X_f = \{0\}$

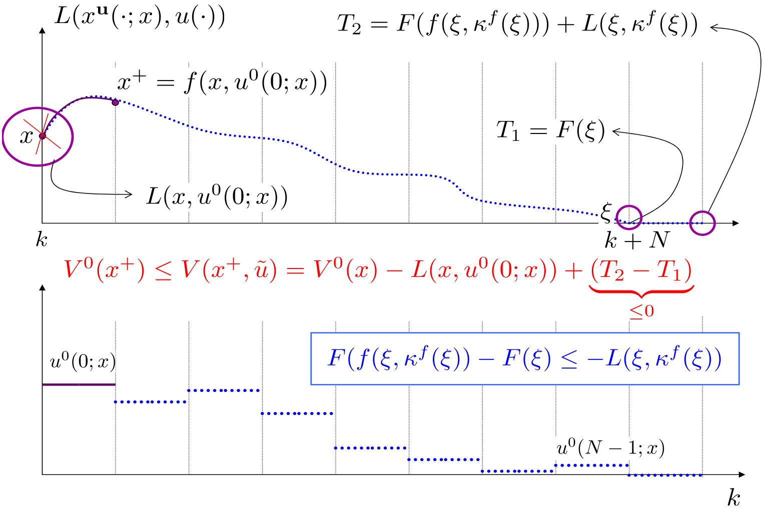
For all $\xi \in X_f$, there exists an admissible control $\kappa^f(\xi)$ such that :

$$f(\xi, \kappa^f(\xi)) \in X_f$$

(X_f is positively invariant under $\kappa^f(\cdot)$)


$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$x^+ = f(x, u^0(0;x))$
x
k
$k+N$
$V^0(x^+) \le V(x^+,\tilde{u}) \le V^0(x) - L(x,u^0(0;x))$
$u^0(0;x)$
$u^0(N-1;x)$
k


$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$T_2 = F(f(\xi,\kappa^f(\xi))) + L(\xi,\kappa^f(\xi))$
$x^+ = f(x,u^0(0;x))$
x
$T_1 = F(\xi)$
$L(x,u^0(0;x))$
ξ
k
$k+N$
$V^0(x^+) \leq V(x^+,\tilde{u}) = V^0(x) - L(x,u^0(0;x)) + (T_2 - T_1)$
$u^0(0;x)$
$u^0(N-1;x)$
k


$L(x^{\mathbf{u}}(\cdot;x),u(\cdot))$
$T_2 = F(f(\xi,\kappa^f(\xi))) + L(\xi,\kappa^f(\xi))$
$x^+ = f(x,u^0(0;x))$
x
$T_1 = F(\xi)$
$L(x,u^0(0;x))$
ξ
k
$k+N$
$V^0(x^+) \leq V(x^+,\tilde{u}) = V^0(x) - L(x,u^0(0;x)) + \underbrace{(T_2 - T_1)}_{\leq 0}$
$u^0(0;x)$
$F(f(\xi,\kappa^f(\xi)) - F(\xi) \leq -L(\xi,\kappa^f(\xi))$
$u^0(N-1;x)$
k

Second useful property of $X_f = \{0\}$

For all $\xi \in X_f$, the admissible control $\kappa^f(\xi)$ is such that :

$$F(f(\xi, \kappa^f(\xi)) - F(\xi) \leq -L(\xi, \kappa^f(\xi))$$

(The terminal cost $F(\cdot)$ is a Lyapunov function for the closed loop dynamics under $\kappa^f(\cdot)$)

First useful property of $X_f = \{0\}$

For all $\xi \in X_f$, there exists an admissible control $\kappa^f(\xi)$ such that :

$$f(\xi, \kappa^f(\xi)) \in X_f$$

(X_f is positively invariant under $\kappa^f(\cdot)$)

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Necessary conditions for well defined and stable NMPC scheme

$$\mathcal{P}(x) \;:\; \min_{\mathbf{u}} \left\{ V(x,\mathbf{u}) \quad \vert \quad \mathbf{u} \in \mathcal{U}(x) \right\} \quad ; \quad x^+ = f(x,u)$$

$$\mathcal{U}(x) := \left\{ \mathbf{u} \in \mathbb{U}^N \mid \forall i \quad x^{\mathbf{u}}(i;x) \in \mathbb{X} \quad \text{and} \quad x^{\mathbf{u}}(N,x) \in X_f \right\}$$

$$V(x,\mathbf{u}) = F(x(N;x)) + \sum_{i=0}^{N} L(x^{\mathbf{u}}(i;x), u(i))$$

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$$V(x,\mathbf{u}) = F(x(N;x)) + \sum_{i=0}^{N} L(x^{\mathbf{u}}(i;x), u(i))$$

Condition 1 : Continuity

The applications f, F, L are continuous in their arguments.

Necessary conditions for well defined and stable NMPC scheme

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$$V(x,\mathbf{u})=F(x(N;x))+\sum_{i=0}^{N}L(x^{\mathbf{u}}(i;x),u(i))$$

Condition 2 : Compactness

- ✓ $\mathbb{U}$ is compact
- ✓ $\mathbb{X}$ and $X_f\subset\mathbb{X}$ are closed.

Necessary conditions for well defined and stable NMPC scheme

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$$V(x,\mathbf{u}) = F(x(N;x)) + \sum_{i=0}^{N} L(x^{\mathbf{u}}(i;x), u(i))$$

Condition 3 : Detectability

The integral cost L must be such that

$$\left\{ L(x,u) \rightarrow 0 \right\} \quad \Rightarrow \quad \left\{ x \rightarrow 0 \right\}$$

Necessary conditions for well defined and stable NMPC scheme

$$\mathcal{P}(x) \; : \; \min_{\mathbf{u}} \left\{ V(x,\mathbf{u}) \quad | \quad \mathbf{u} \in \mathcal{U}(x) \right\} \quad ; \quad x^+ = f(x,u)$$

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Condition 4 : X_f is positively invariant under some local $\kappa^f(\cdot)$

For all $\xi \in X_f$, there exists an admissible control $\kappa^f(\xi)$ such that :

$$f(\xi, \kappa^f(\xi)) \in X_f$$

Necessary conditions for well defined and stable NMPC scheme

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$$V(x,\mathbf{u})=F(x(N;x))+\sum_{i=0}^{N}L(x^{\mathbf{u}}(i;x),u(i))$$

Condition 5 : The terminal cost $F(\cdot)$ is a Lyapunov under $\kappa^f(\cdot)$

For all $\xi\in X_f$, the admissible control $\kappa^f(\xi)$ is such that :

$$F(f(\xi,\kappa^f(\xi))-F(\xi)\leq -L(\xi,\kappa^f(\xi))$$

Necessary conditions for well defined and stable NMPC scheme

$$\mathcal{P}(x)\ :\ \min_{\mathbf{u}}\left\{V(x,\mathbf{u})\quad\middle|\quad \mathbf{u}\in\mathcal{U}(x)\right\}\quad ;\quad x^+=f(x,u)$$

$$\mathcal{U}(x):=\left\{\mathbf{u}\in\mathbb{U}^N \mid \forall i\quad x^{\mathbf{u}}(i;x)\in\mathbb{X}\quad \text{and}\quad x^{\mathbf{u}}(N,x)\in X_f\right\}$$

$$V(x,\mathbf{u})=F(x(N;x))+\sum_{i=0}^{N}L(x^{\mathbf{u}}(i;x),u(i))$$

Condition 6 : Feasibility

There is at least a sequence that meets the constraints (in particular $x(0)\in X_N$ (the subset of state steerable in N steps to X_f with bounded controls in $\mathbb{U}$).

To summarize

The system $x^+ = f(x,u)$

The cost function $F(x(N)) + \sum_{i=0}^{N} L(x(i),u(i))$

---

1) f, L, L are continuous.
2) $\mathbb{U}$ is compact, $\mathbb{X}$ and $X_f \subset \mathbb{X}$ are closed.
3) $\{L(x,u) \to 0\} \Rightarrow \{x \to 0\}$
4) X_f is positively invariant (with some $\kappa_f(\cdot)$.
5) $F(\cdot)$ is a Lyapunov function under $\kappa_f(\cdot)$.
6) $x(0) \in X_N$.

How to choose X_f and $\kappa_f(\cdot)$?

1. Infinite horizon $N = \infty$.

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3. If the linearized system at 0 is stabilizable, then
   - $\kappa_f(x) = -Kx$

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   - $\kappa_f(x) = -Kx$
   - $(A - BK)^T P(A - BK) - P = -Q$ for some $P, Q > 0$

How to choose X_f and $\kappa_f(\cdot)$?

1. Infinite horizon $N = \infty$.
2. Point-wise final constraint $X_f = \{0\}$.
3. If the linearized system at 0 is stabilizable, then
   - $\kappa_f(x) = -Kx$
   - $(A - BK)^T P(A - BK) - P = -Q$ for some $P, Q > 0$
   - X_f is a level set of $x^T Px$, namely

$$X_f = \left\{x \quad | \quad x^T Px \leq \varrho\right\}$$

   for a sufficiently small $\varrho > 0$.

Example

$$\begin{aligned} x_1^+ &= x_1 + (1 + x_2^2)u \\ x_2^+ &= \frac{3}{2}x_2 - x_1 e^u \end{aligned}$$

- ✓ Open-loop instable.
- ✓ The set of equilibrium states is given by

$$\mathcal{E}_{st} = \left\{ x^{(\alpha)} := \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} \quad ; \quad \alpha \in \mathbb{R} \right\}$$

Control objective starting at $x^{(0)} = (0, 0)$, stabilize the system around $x^{(1)} = (1, 2)$.

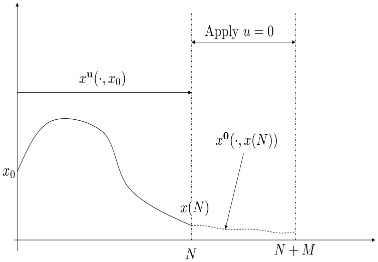
Consider the cost function

$$V(x, \mathbf{u}) := F(x(N)) + \sum_{i=0}^{N} L(x(i), u(i))$$

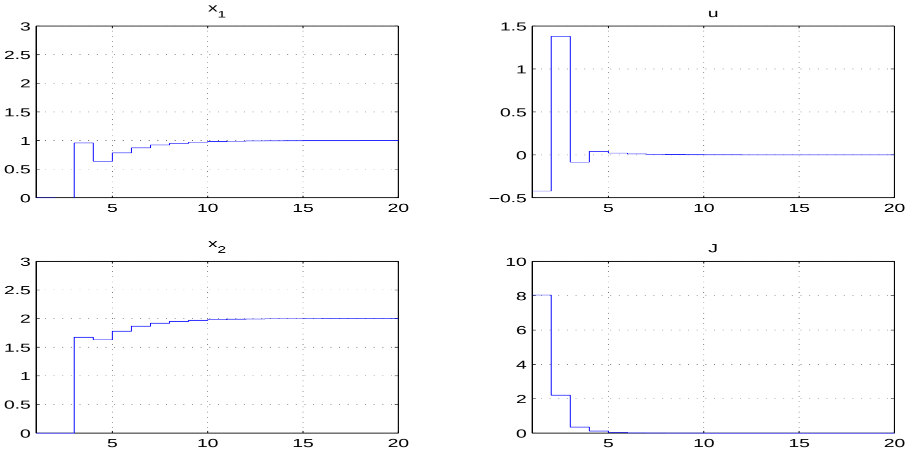
where

$$L(x, u) := \|x - x^{(1)}\|^2 + ru^2$$

$$F(x) = \sum_{i=1}^{M-N} x^{\mathbf{0}}(i; x)$$



Test 1 : $N = 2$, $M = 2$, $r = 1$



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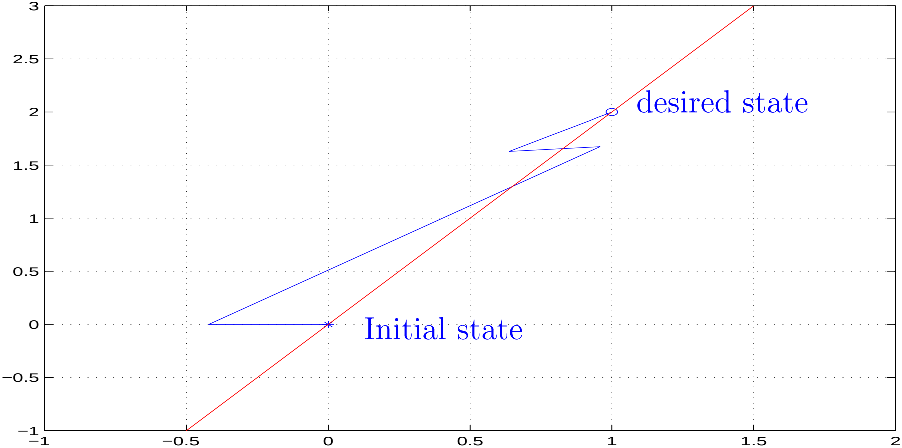
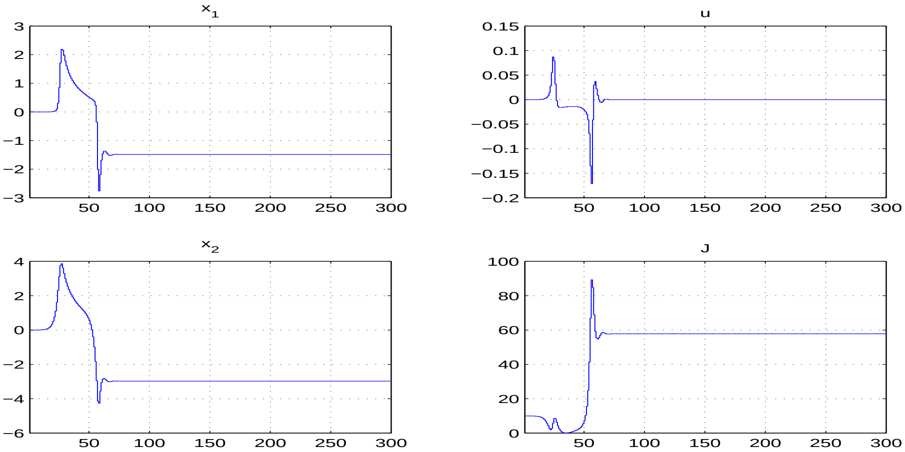


Let us assume that one looks for solution such that $u \leq 0.1$

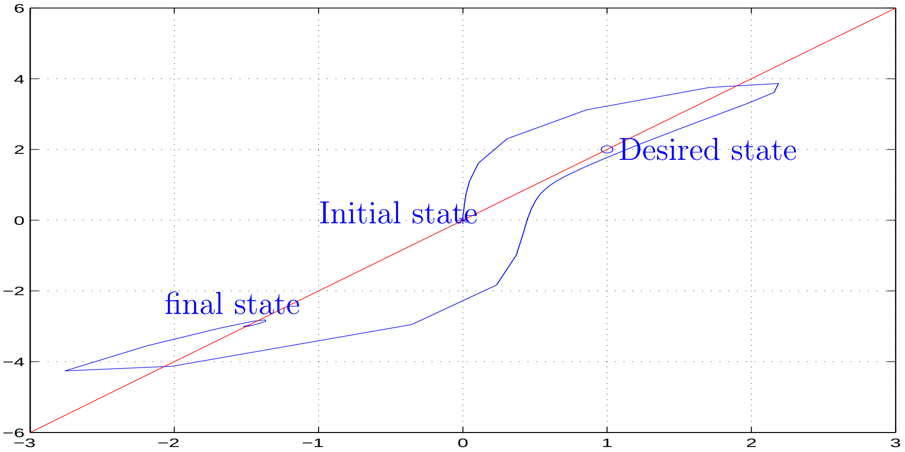
Assume that for that reason one takes

$$r = 160$$

Test 2 : $N = 2$, $M = 2$, $r = 160$

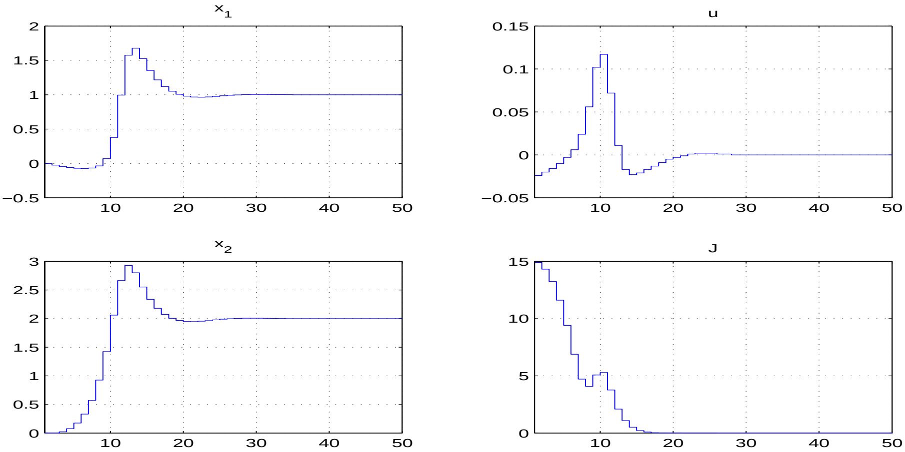


Test 2 : $N = 2$, $M = 2$, $r = 160$

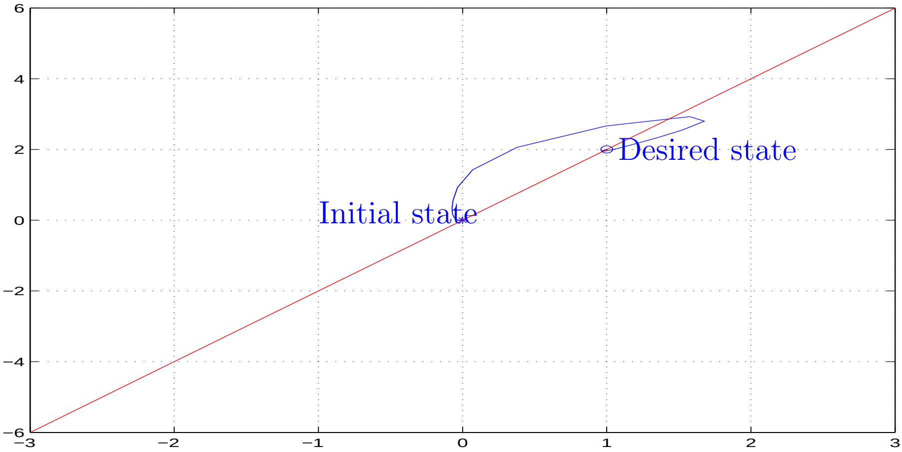


Let us take $M = 3$ (instead of 2)

Test 3 : $N = 2$, $M = 3$, $r = 160$



Test 3 : $N = 2$, $M = 3$, $r = 160$



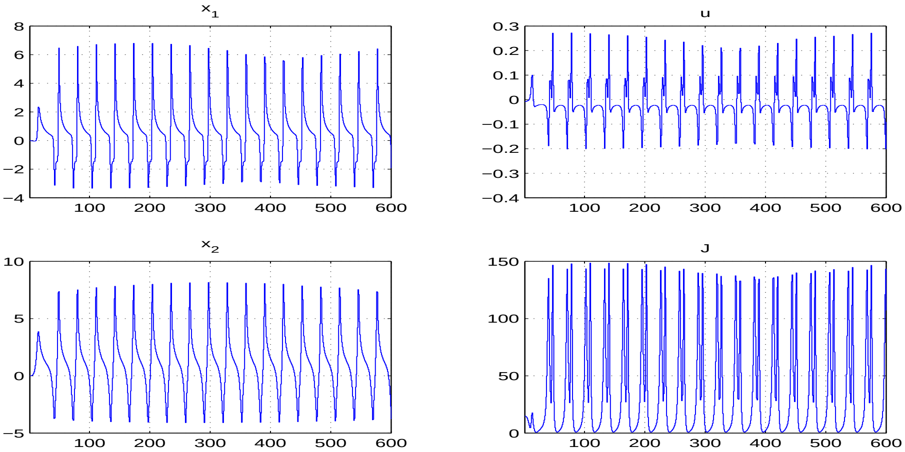
Increasing M enabled the target state to be reached.

However, the control is again above 0.1,

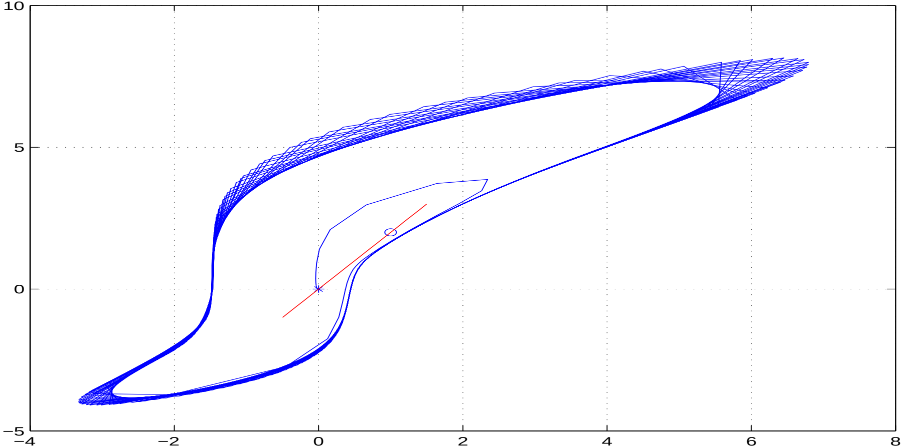
So let us increase r again by taking

$$r = 500$$

Test 4 : $N = 2$, $M = 3$, $r = 500$



Test 4 : $N = 2$, $M = 3$, $r = 500$



A limit cycle appears

and the control is beyond 0.25 ! ! !

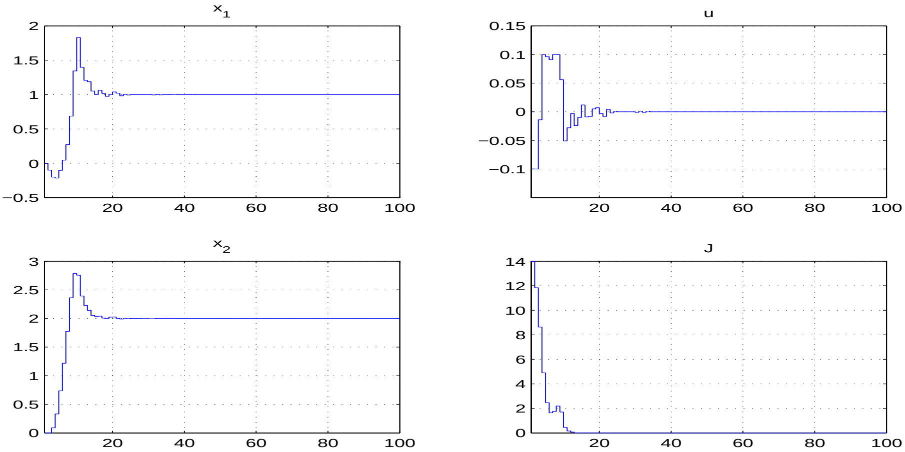
Let us explicitly impose the constraint

$$\mathbb{U} = [-0.1, 0.1]$$

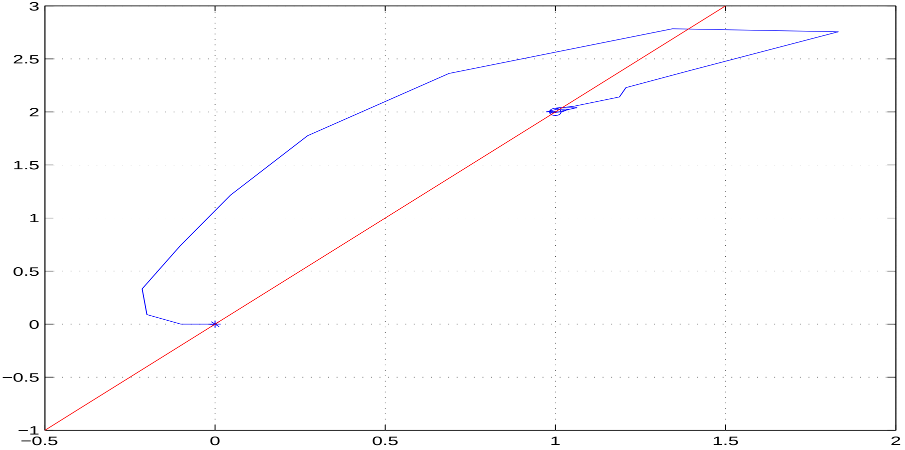
and test it with the configuration

$$r = 1 \quad ; \quad N = M = 3$$

Test 5 : $N = 3$, $M = 3$, $r = 1$, explicit constraint $|u| \leq 0.1$



Test 5 : $N = 3$, $M = 3$, $r = 1$, explicit constraint $|u| \leq 0.1$



The example shows that

- Nonlinear Model Predictive Control is a "*generic*" solution.

  (Who remember the system's equations ? ! !)

```
subroutine syst(x,u,xplus)
    implicit none
    double precision :: x(2), u, xplus(2)
    xplus(1)=x(1)+(1.0d0+x(2)**2)*u
    xplus(2)=1.5*x(2)-x(1)*dexp(u)
end subroutine syst
```

```
subroutine syst_long(x0,nu,ns,utilde,xtilde)
   implicit none
   external syst
   integer :: nu, ns, i
   double precision :: x0(2), utilde(nu), xtilde(ns+1,2)
   xtilde(1,:)=x0
   do i=1,nu
      call syst(xtilde(i,:),utilde(i),xtilde(i+1,:))
   enddo
   do i=nu+1,ns
      call syst(xtilde(i,:),0.0d0,xtilde(i+1,:))
   enddo
end subroutine syst_long
```

```
subroutine critere(nu,utilde,J)
   use param
   use imsl
   implicit none
   integer :: nu, i
   double precision :: utilde(nu), J, xtilde(M+1,2)
   double precision :: dx(2)
   call syst_long(x0_glob,nu,M,utilde,xtilde)
   J=0.d0
   do i=1,nu
      J=J+qu*dabs(utilde(i))**2
      dx = xtilde(i+1,:)-xd_glob
      J=J+qx*dot_product(dx,eye(2).x.dx)
   enddo
   do i=nu+1,M
      dx = xtilde(i+1,:)-xd_glob
      J = J+qx*dot_product(dx,eye(2).x.dx)
   enddo
end subroutine critere
```

Computation of the optimal sequence

```
subroutine uhat(x0,xd,nu,utilde,J)
    use param
    implicit none
    external critere
    integer :: nu, maxfcn=100
    double precision :: x0(2), xd(2), utilde(nu), J
    double precision :: utilde_guess(nu), ulb(nu), uub(nu)
    x0_glob=x0; xd_glob=xd; utilde_guess=0.0d0
    ulb=-umax; uub=umax
    call dbcpol(critere,nu,utilde_guess,0,ulb,uub,1.0d-8,maxfcn,utilde,J)
end subroutine uhat
```

The example shows that

- Nonlinear Model Predictive Control is a "*generic*" solution.

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- Nonlinear Model Predictive Control is a "*generic*" solution.
- Easy handling of constraints
- The stability IS AN ISSUE

To summarize

The system $x^+ = f(x,u)$

The cost function $F(x(N)) + \sum_{i=0}^{N} L(x(i),u(i))$

---

1) f, L, L are continuous.
2) $\mathbb{U}$ is compact, $\mathbb{X}$ and $X_f \subset \mathbb{X}$ are closed.
3) $\{L(x,u) \to 0\} \Rightarrow \{x \to 0\}$
4) X_f is positively invariant (with some $\kappa_f(\cdot)$.
5) $F(\cdot)$ is a Lyapunov function under $\kappa_f(\cdot)$.
6) $x(0) \in X_N$.

This general result summarizes 90% of existing works on the stability of the NMPC schemes.

This does not include the contractive schemes ...

A preliminary example

Consider the nonlinear system

$$\dot{x}_1 = x_2 \quad ; \quad \dot{x}_2 = x_1 u$$

and the "*candidate*" Lyapunov function

$$V(x) = \frac{1}{2}\left[x_1^2 + x_2^2\right]$$

Compute the derivative of V

$$\dot{V}(x) = x_1 x_2 (1 + u)$$

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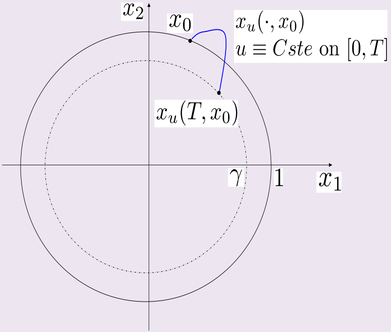
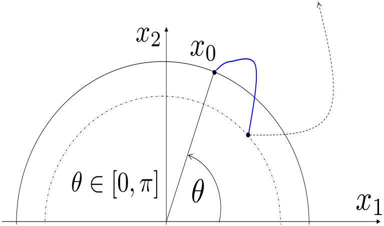
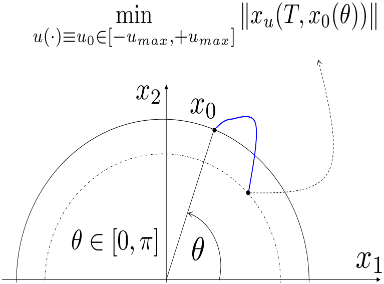
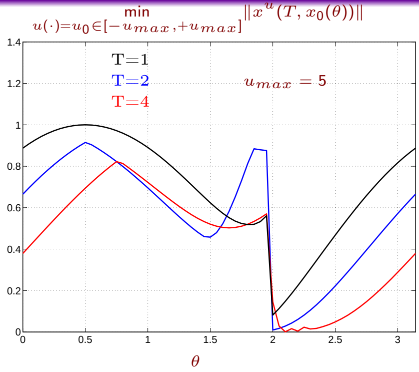
But...

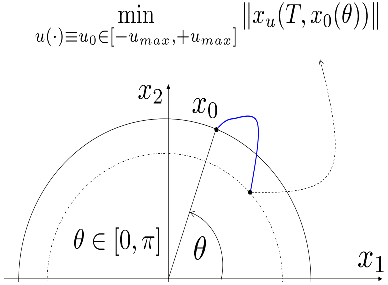


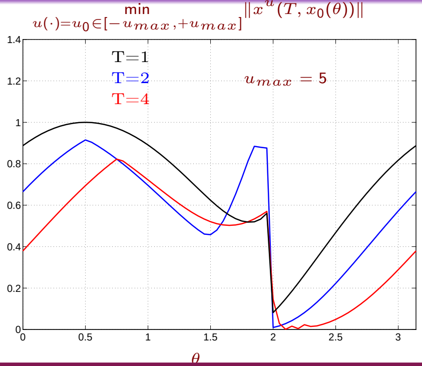
$$\min_{u(\cdot)\equiv u_0\in[-u_{max},+u_{max}]} \|x_u(T,x_0(\theta))\|$$




$\min_{u(\cdot)\equiv u_0\in[-u_{max},+u_{max}]} \|x_u(T,x_0(\theta))\|$
x_2
x_0
$\theta\in[0,\pi]$
θ
x_1


$\min_{u(\cdot)=u_0\in[-u_{max},+u_{max}]} \|x^u(T,x_0(\theta))\|$
T=1
T=2
T=4
$u_{max}=5$
θ





To summarize : A long term contraction property

Whatever is the initial state $x_0 \in B(0,1)$, there exists constant control $u \in [-5,+5]$ such that

$$V\big(x^u(2,x_0)\big)\| \leq 0.9\ V(x_0) \quad \text{or} \quad \Big[V\big(x^u(4,x_0)\big)\| \leq 0.82\ V(x_0)\Big]$$

A bad formulation

Define a receding horizon feedback based on the following open-loop optimization problem

$$\min_{\mathbf{u}(\cdot),\Delta\in[0,T]} \int_0^{\Delta} L\Big(x_u(\tau,x(t))\Big)d\tau \quad \text{under } V(x_u(t+\Delta,x(t))) \leq \gamma\, V(x(t))$$

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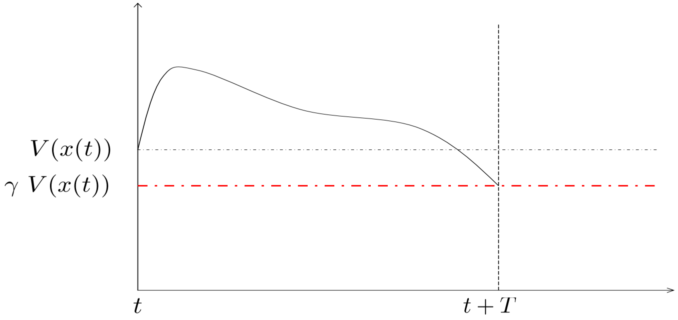

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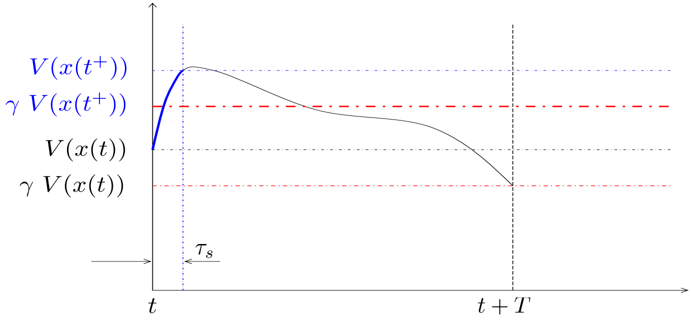

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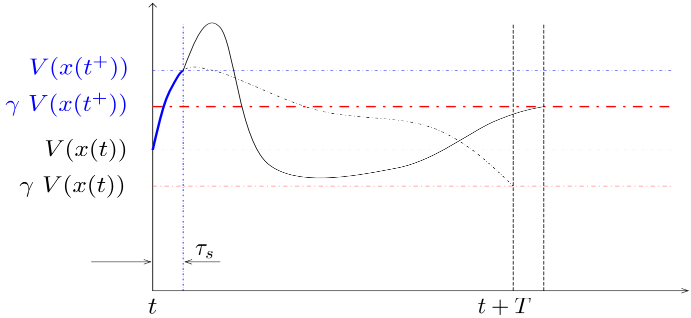

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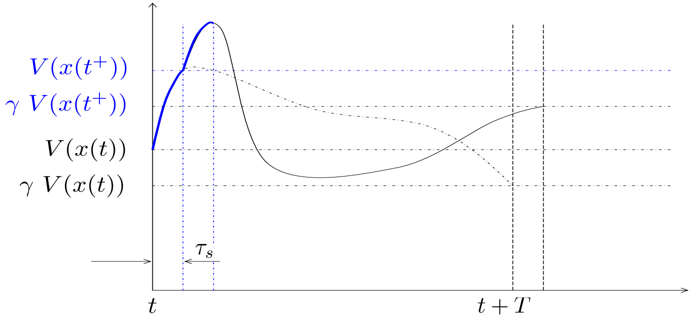

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Updating systematically the contractive constraint

$$V(x_u(\Delta,x(t))) \leq \gamma\, V(x(t))$$

May cause instability in closed-loop

Kothare, S. L. de Oliveira and Morari, M. IEEE-TAC Vol 45 pp 1053-1071 (2000)

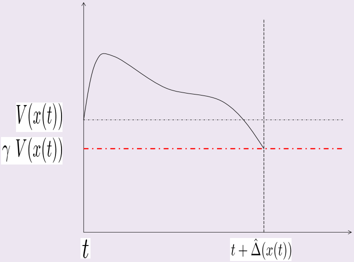
Kothare, S. L. de Oliveira and Morari, M. IEEE-TAC Vol 45 pp 1053-1071 (2000)

Either

Use the open-loop control

$$\hat{u}(\cdot, x(t))$$

on $[t, t + \hat{\Delta}(x(t))]$



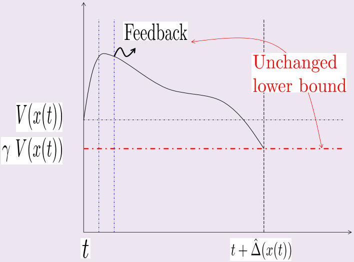
Kothare, S. L. de Oliveira and Morari, M. IEEE-TAC Vol 45 pp 1053-1071 (2000)

Or

Memorize $x(t)$ and use

$$V(x_u(\Delta, x(t+k\tau_s))) \leq \gamma\, V(x(t))$$

in a RH scheme during the time interval $[t, t+\hat{\Delta}(x(t))]$



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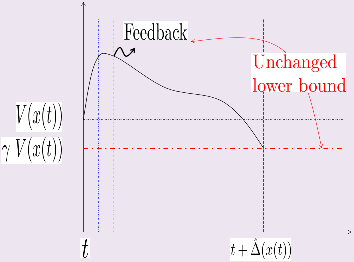

- Non standard RH implementation
- Lack of reactivity
- Potential feasibility problems
  - In presence of disturbances
  - Under truncated optimization

Consider nonlinear systems

$$\dot{x} = f(x,u) \quad ; \quad x \in \mathbb{R}^n \quad ; \quad u \in \mathbb{R}^m \quad ; \quad f \text{ continuous}$$

satisfying the following assumption

Infinitely fast state excursions need infinite control

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Infinitely fast state excursions need infinite control

For all finite horizon $T > 0$,

$$\lim_{\|x_0\| \to \infty} \left[\min_{\mathbf{u} \in \mathbb{W}^{[0,T]}} \quad \min_{t \in [0,T]} \|F(t, x_0, \mathbf{u})\| \right] = \infty$$

for all compact subset $\mathbb{W} \subset \mathbb{R}^m$. $\flat$

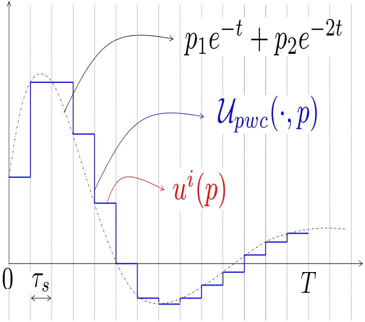
- ✓ Choose a sampling period τ_s
- ✓ Define a τ_s-piece-wise constant control profile

$$\mathcal{U}_{pwc}(\cdot, p) \quad ; \quad p \in \mathbb{P}$$

- ✓ The parametrization is called "*translatable*" if for all $p \in \mathbb{P}$, there is $p^+ \in \mathbb{P}$ s.t.

$$u^i(p^+) = u^{i+1}(p)$$

$$\forall i \in \{1, \dots, N-1\}$$



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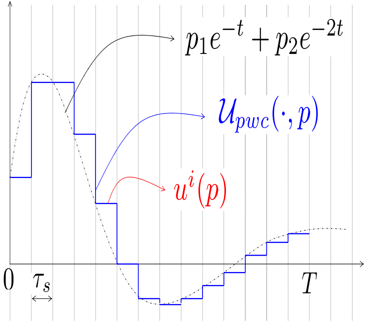

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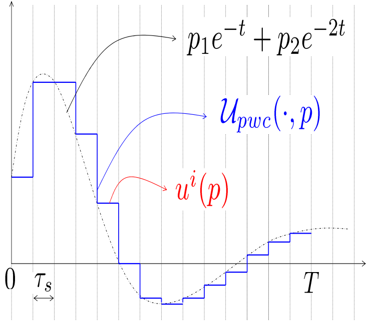

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Notation $F(\cdot, x, p)$, $V(\cdot, x, p)$

$$p^+ = \begin{pmatrix} e^{-\tau_s} & 0 \\ 0 & e^{-2\tau_s} \end{pmatrix} p$$

The strong contraction property

1. $\exists \gamma \in]0,1[$ s.t. for all x, there exists $p^c(x) \in \mathbb{P}$ such that

$$\min_{q \in \{1,\dots,N\}} V(q\tau_s, x, p^c(x)) \leq \gamma V(x)$$

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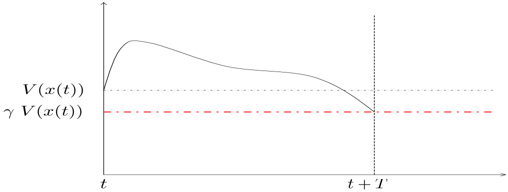
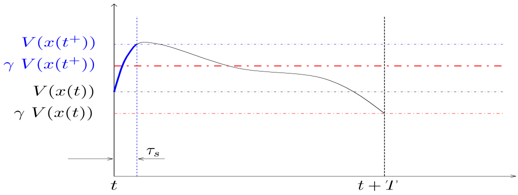
2. $p_c(\cdot)$ is bounded over bounded sets
3. $\exists$ a continuous function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ s.t. for all x :

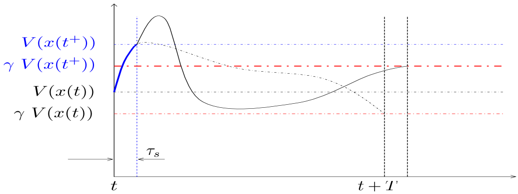
$$\|V_{1\rightarrow N}(\cdot, x, p^c(x))\|_\infty \leq \varphi(x) \cdot V(x)$$

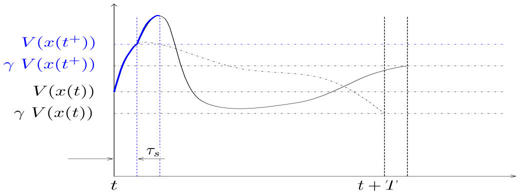
where

$$\|V_{1\rightarrow q}(\cdot, x, p)\|_\infty = \max_{i\in\{1,\dots,q\}} V(i\tau_s, x, p)$$


$V(x(t))$
$\gamma\ V(x(t))$
t
$t+T$


$V(x(t^+))$
$\gamma\ V(x(t^+))$
$V(x(t))$
$\gamma\ V(x(t))$
τ_s
t
$t+T$


$V(x(t^+))$
$\gamma\ V(x(t^+))$
$V(x(t))$
$\gamma\ V(x(t))$
τ_s
t
$t+T$


$V(x(t^+))$
$\gamma\ V(x(t^+))$
$V(x(t))$
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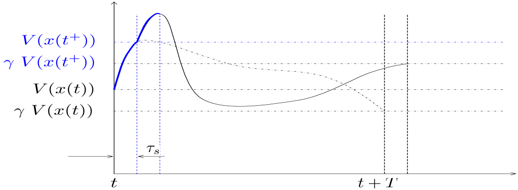
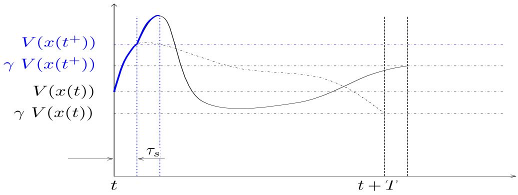
τ_s
t
$t+T$



The open-loop optimal control problem

$$\min_{(q,p)\in\{1,\dots,N\}\times\mathbb{P}_{\mathbb{X}}} \quad V(q\tau_s,x,p)+\alpha\frac{q}{N}\cdot\min\{\varepsilon^2,\|V_{1\to q}(\cdot,x,p)\|_\infty\}$$



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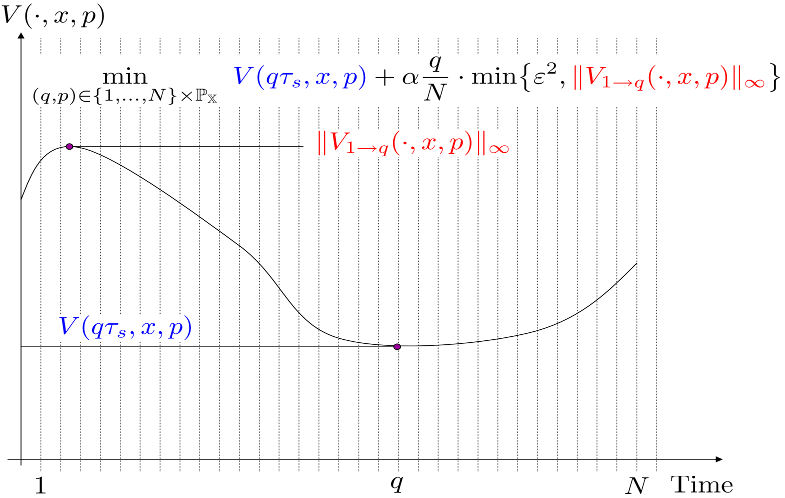
The receding-horizon state feedback

$$u(k\tau_s+\tau)=u^1(\hat{p}(x(k\tau_s)))\quad\forall\tau\in[0,\tau_s[$$


$V(\cdot, x, p)$
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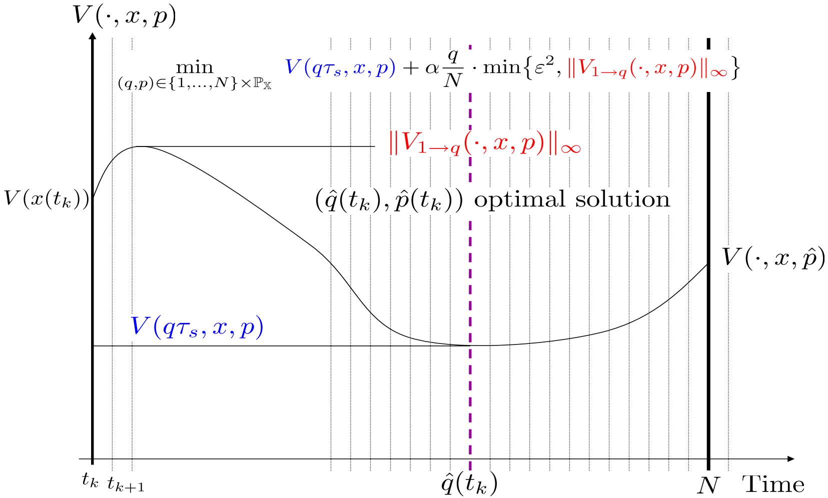
$\|V_{1\to q}(\cdot, x, p)\|_\infty$
$V(q\tau_s, x, p)$
1
q
N
Time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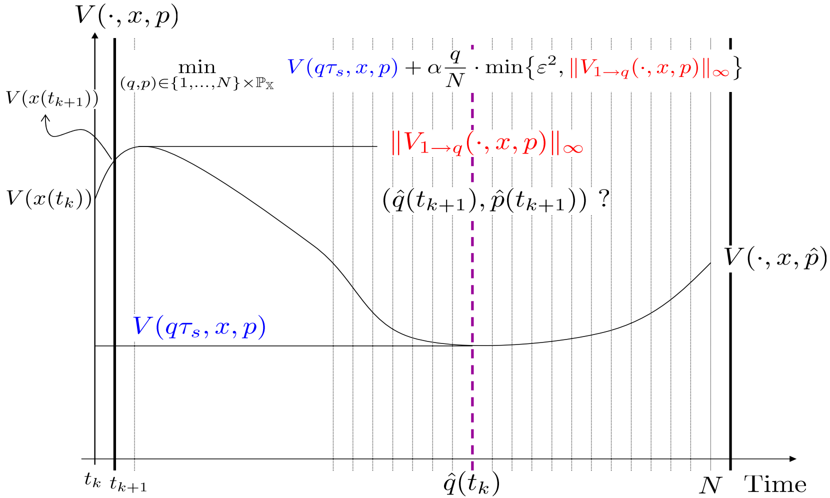
$\|V_{1\to q}(\cdot, x, p)\|_\infty$
$V(x(t_k))$
$(\hat{q}(t_k), \hat{p}(t_k))$ optimal solution
$V(\cdot, x, \hat{p})$
$V(q\tau_s, x, p)$
t_k t_{k+1}
$\hat{q}(t_k)$
N Time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$V(x(t_{k+1}))$
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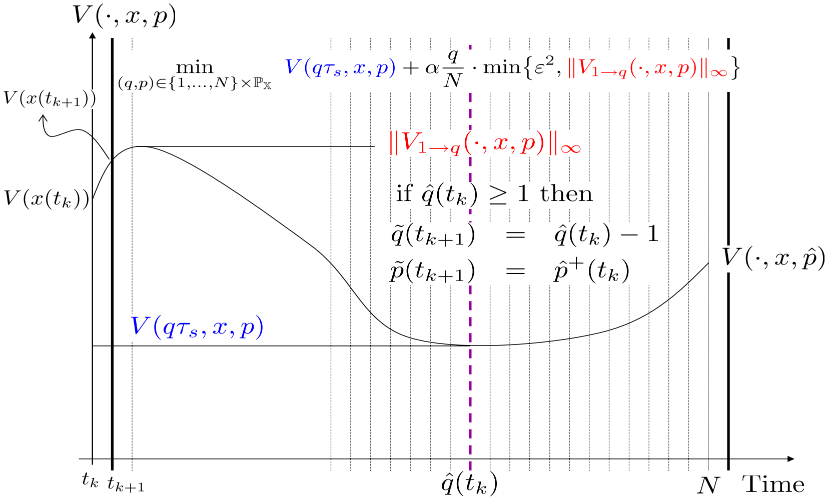
$V(x(t_k))$
$(\hat{q}(t_{k+1}), \hat{p}(t_{k+1}))$?
$V(\cdot, x, \hat{p})$
$V(q\tau_s, x, p)$
t_k t_{k+1}
$\hat{q}(t_k)$
N Time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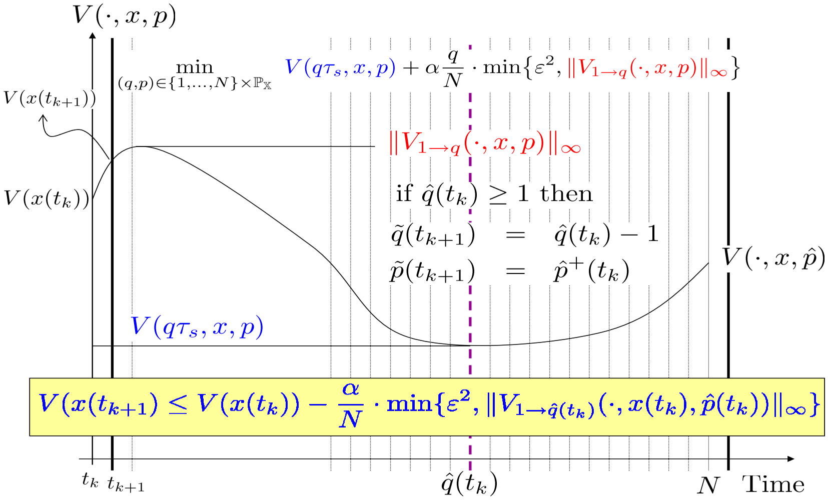
$V(x(t_{k+1}))$
$\|V_{1\to q}(\cdot, x, p)\|_\infty$
if $\hat{q}(t_k) \geq 1$ then
$V(x(t_k))$
$\tilde{q}(t_{k+1}) = \hat{q}(t_k) - 1$
$\tilde{p}(t_{k+1}) = \hat{p}^+(t_k)$
$V(\cdot, x, \hat{p})$
$V(q\tau_s, x, p)$
t_k t_{k+1}
$\hat{q}(t_k)$
N Time



$$V(x(t_{k+1}) \leq V(x(t_k)) - \frac{\alpha}{N}\cdot\min\{\varepsilon^2, \|V_{1\to\hat{q}(t_k)}(\cdot, x(t_k), \hat{p}(t_k))\|_\infty\}$$

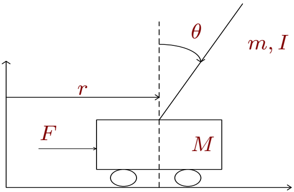
Basic Result

If the following conditions hold

1. Continuity (system/parametrization)
2. Infinitely fast excursions need infinite controls
3. The control parametrization is translatable on

$$\mathbb{P}_{\mathbb{X}} := \mathbb{P} \cap B\left(0, \sup_{x \in \bar{B}(0,\rho(\mathbb{X}))} \|p^c(x)\| + \varepsilon_0\right) \subseteq \mathbb{P} \subseteq \mathbb{R}^{n_p}$$

Then, $\exists$ sufficiently small $\varepsilon > 0$ and $\alpha > 0$ such that the RH feedback is well defined and makes the origin $x = 0$ asymptotically stable for the resulting CL dynamics with a region of attraction that contains $\mathbb{X}$.

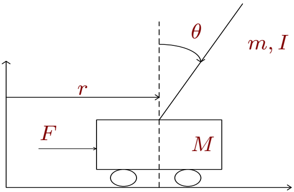


The system equations

$$\begin{pmatrix} mL^2 + I & mL\cos\theta \\ mL\cos\theta & m+M \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{r} \end{pmatrix} = \begin{pmatrix} mLg\sin\theta - k_\theta\dot{\theta} \\ F + mL\dot{\theta}^2\sin\theta - k_x\dot{r} \end{pmatrix}$$

A pre-compensator

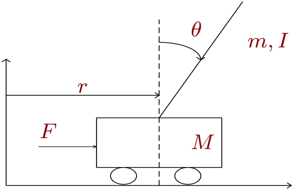
$$F = -K_{pre}\begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u$$



The system equations

$$\dot{x}_1 = x_3 \quad ; \quad \dot{x}_2 = x_4$$

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = [M(x)]^{-1} \begin{pmatrix} mLg\sin(x_1) - k_\theta \cdot x_3 \\ -K_{pre_1}x_2 - K_{pre_2}x_4 + mLx_3^2\sin(x_1) - k_x x_4 + u \end{pmatrix}$$

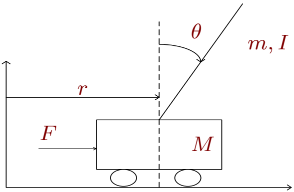


Control parametrization

$$u^i(p) = p \cdot e^{-t_i/t_r} \quad ; \quad t_i = \frac{(i-1)\tau_s}{N}$$

where $p \in \mathbb{P}(x) := [p_{min}(x), p_{max}(x)]$ s.t

$$\begin{aligned} p_{min}(x) &= -F_{max} + K_{pre_1} x_2 + K_{pre_2} x_4 \\ p_{max}(x) &= +F_{max} + K_{pre_1} x_2 + K_{pre_2} x_4 \end{aligned}$$



Use the contractive RH formulation given by :

$$V(x) = \frac{1}{2}\left[\dot{\theta}^2 + \beta r^2 + \dot{r}^2\right] + [1 - \cos(\theta)]^2$$

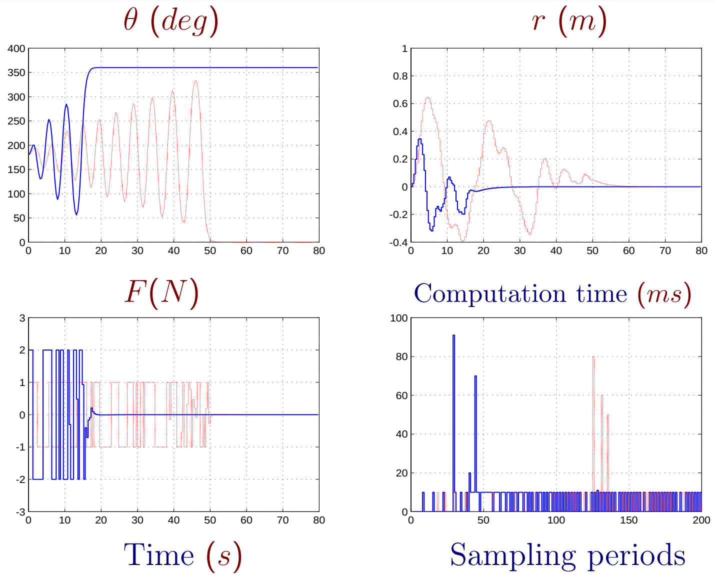
$$\min_{(q,p)\in\{1,\dots,N\}\times\mathbb{P}(x)} \quad V(q\tau_s, x, p) + \frac{\alpha}{N}\cdot\min\{\varepsilon, \|V_{1\to q}(\cdot, x, p)\|_\infty\}$$

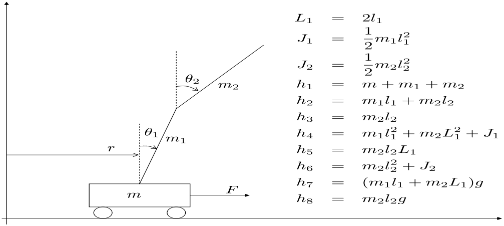
$$u(k\tau_s + \tau) = u^1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s[$$

The parameters of the controller

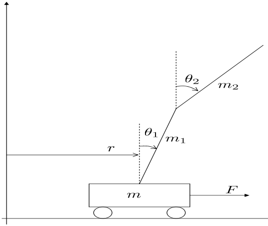
parameter	value	signification
τ_s	$0.4\ s$	sampling period
N	8	horizon length
t_r	0.2	Constant for the control param.
$\alpha = \varepsilon$	0.01	cost function parameters
K_{pre}	$\begin{pmatrix} 2.5 & 10.0 \end{pmatrix}$	Pre-compensation gain
F_{max}	$\in \{1.0, 2.0\}$	saturation level on F
β	$\in 10$	weighting coefficient on r

Runs on a $1.3\ GHz$ Pentium-III


θ (deg)
r (m)
$F(N)$
Computation time (ms)
Time (s)
Sampling periods



$$
\begin{aligned}
L_1 &= 2l_1 \\
J_1 &= \frac{1}{2}m_1 l_1^2 \\
J_2 &= \frac{1}{2}m_2 l_2^2 \\
h_1 &= m + m_1 + m_2 \\
h_2 &= m_1 l_1 + m_2 l_2 \\
h_3 &= m_2 l_2 \\
h_4 &= m_1 l_1^2 + m_2 L_1^2 + J_1 \\
h_5 &= m_2 l_2 L_1 \\
h_6 &= m_2 l_2^2 + J_2 \\
h_7 &= (m_1 l_1 + m_2 L_1)g \\
h_8 &= m_2 l_2 g
\end{aligned}
$$



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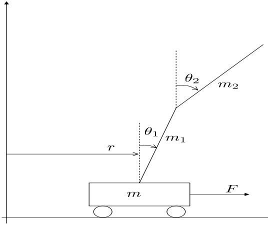
h_6 &= m_2 l_2^2 + J_2 \\
h_7 &= (m_1 l_1 + m_2 L_1) g \\
h_8 &= m_2 l_2 g
\end{aligned}
$$

System equations

$$h_1\ddot{r} + h_2\ddot{\theta}_1\cos\theta_1 + h_3\ddot{\theta}_2\cos\theta_2 = h_2\dot{\theta}_1^2\sin\theta_1 + h_3\dot{\theta}_2^2\sin\theta_2 + F$$

$$h_2\ddot{r}\cos\theta_1 + h_4\ddot{\theta}_1 + h_5\ddot{\theta}_2\cos(\theta_1-\theta_2) = h_7\sin\theta_1 - h_5\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

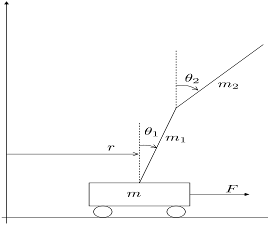
$$h_3\ddot{r}\cos\theta_2 + h_5\ddot{\theta}_1\cos(\theta_1-\theta_2) + h_6\ddot{\theta}_2 = h_5\dot{\theta}_1^2\sin(\theta_1-\theta_2) + h_8\sin\theta_2$$



$$
\begin{aligned}
L_1 &= 2l_1\\
J_1 &= \frac{1}{2}m_1 l_1^2\\
J_2 &= \frac{1}{2}m_2 l_2^2\\
h_1 &= m + m_1 + m_2\\
h_2 &= m_1 l_1 + m_2 l_2\\
h_3 &= m_2 l_2\\
h_4 &= m_1 l_1^2 + m_2 L_1^2 + J_1\\
h_5 &= m_2 l_2 L_1\\
h_6 &= m_2 l_2^2 + J_2\\
h_7 &= (m_1 l_1 + m_2 L_1) g\\
h_8 &= m_2 l_2 g
\end{aligned}
$$

Pre-compensation

$$F = -K_{pre} \cdot \begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u$$



$$
\begin{aligned}
L_1 &= 2l_1 \\
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h_1 &= m + m_1 + m_2 \\
h_2 &= m_1 l_1 + m_2 l_2 \\
h_3 &= m_2 l_2 \\
h_4 &= m_1 l_1^2 + m_2 L_1^2 + J_1 \\
h_5 &= m_2 l_2 L_1 \\
h_6 &= m_2 l_2^2 + J_2 \\
h_7 &= (m_1 l_1 + m_2 L_1) g \\
h_8 &= m_2 l_2 g
\end{aligned}
$$

Control parametrization

$$u^i(p) = p_1 \cdot e^{\lambda_1 t_i} + p_2 e^{-\lambda_2 t_i} \quad ; \quad t_i = \frac{(i-1)\tau_s}{N}$$

$$p_{min}(x) := \frac{1}{2}\left[-F_{max} + K_{pre}\begin{pmatrix} r \\ \dot{r} \end{pmatrix}\right] \quad ; \quad p_{max}(x) := \frac{1}{2}\left[+F_{max} + K_{pre}\begin{pmatrix} r \\ \dot{r} \end{pmatrix}\right]$$

The contractive RH controller

$$
\begin{aligned}
V(x) &= \frac{h_4}{2}\dot{\theta}_1^2 + \frac{h_6}{2}\dot{\theta}_2^2 + h_5\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + h_7\big[1-\cos(\theta_1)\big] + \\
&+ h_8\big[1-\cos(\theta_2)\big] + h_1\big[r^2+\dot{r}^2\big] \\
u(k\tau_s+t) &= K_{RH}(x(k\tau_s)) := u^1(\hat{p}(x(k\tau_s))) \quad ; \quad t\in[0,\tau_s[
\end{aligned}
$$

The contractive RH controller

$$\begin{aligned} V(x) &= \frac{h_4}{2}\dot{\theta}_1^2 + \frac{h_6}{2}\dot{\theta}_2^2 + h_5\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + h_7\big[1-\cos(\theta_1)\big] + \\ &+ h_8\big[1-\cos(\theta_2)\big] + h_1\big[r^2+\dot{r}^2\big] \\ u(k\tau_s+t) &= K_{RH}(x(k\tau_s)) := u^1(\hat{p}(x(k\tau_s))) \quad ; \quad t\in[0,\tau_s[\end{aligned}$$

A local LQR controller

$$K_L(x) = -L\cdot\begin{pmatrix} x_1^m \\ x_2^m \\ x_3 \\ \vdots \\ x_6 \end{pmatrix}$$

solving the discrete time Riccati equation

$$A_d^T S A_d - S - (A_d^T S B_d)(R + B_d^T S B_d^T)(B_d^T S A_d) + Q = 0$$

Hybrid controller for swing-up and stabilization of the double inverted pendulum

To summarize, the hybrid controller is given by

$$u(k\tau_s + \tau) = \begin{cases} K_{RH}(x(k\tau_s)) & \text{if} \quad \|x(k\tau_s)\|_S^2 > \eta \\ K_L(x(k\tau_s)) & \text{otherwise} \end{cases}$$

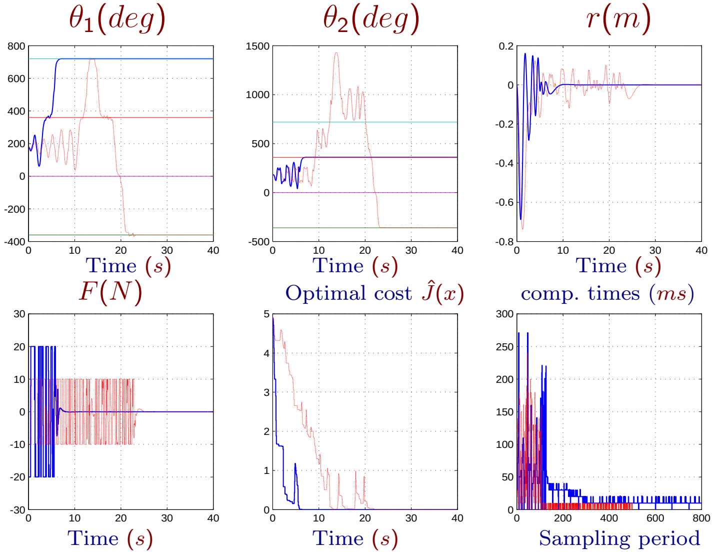
Hybrid controller for swing-up and stabilization of the double inverted pendulum

To summarize, the hybrid controller is given by

$$u(k\tau_s + \tau) = \begin{cases} K_{RH}(x(k\tau_s)) & \text{if} \quad \|x(k\tau_s)\|_S^2 > \eta \\ K_L(x(k\tau_s)) & \text{otherwise} \end{cases}$$

The parameters of the controller

parameter	value	signification
τ_s	0.3 s	sampling period
N	10	horizon length
L	$(360, 29)$	(linear controller gain)
(λ_1, λ_2)	$(100, 20)$	Control parametrization
η	1.0	switching threshold
i_{max}	20	Max number of function evaluation


θ1(deg)
Time (s)
θ2(deg)
Time (s)
r(m)
Time (s)
F(N)
Time (s)
Optimal cost Ĵ(x)
Time (s)
comp. times (ms)
Sampling period

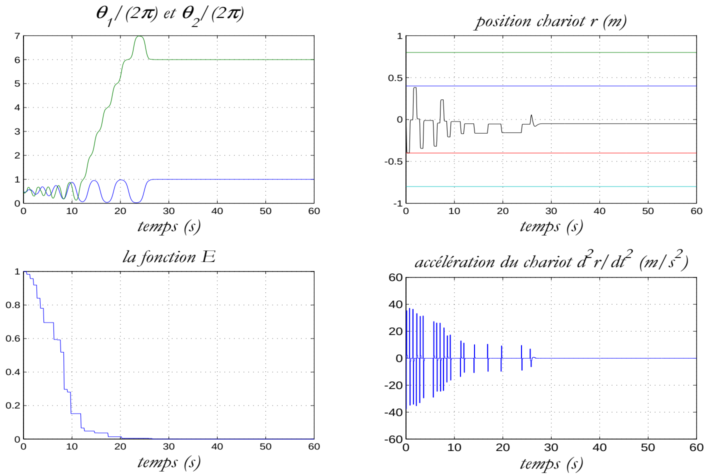
Double inverted pendulum

le film

The twin pendulum

Movie

The twin pendulum



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